

UNIVERSITÀ DEGLI STUDI DI ROMA 3

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**5D DIFFERENTIAL CALCULUS AND NOETHER ANALYSIS  
OF TRANSLATION SYMMETRIES  
IN  $\kappa$ -MINKOWSKI NONCOMMUTATIVE SPACETIME**

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**Abstract**

We perform a Noether analysis for a description of translation transformations in 4D  $\kappa$ -Minkowski noncommutative spacetime which is based on the structure of a 5D differential calculus. Taking properly into account the properties of the differential calculus we arrive at an explicit formula for the conserved charges. We also propose a choice of basis for the 5D calculus which leads to an intuitive description of time derivatives.

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# Introduction

Various arguments suggest that our current description of particle physics would require a profound revision in order to describe processes at the Planck scale  $E_p$ , defined as

$$E_p = \sqrt{\frac{\hbar c}{G}} \simeq 10^{19} GeV.$$

At such high energy both quantum and gravitational effects are important, and the Standard Model of particle physics appears to be incomplete since it neglects gravity.

A large research effort has been devoted to the search for a “Quantum Gravity”, i.e. a theory giving a unified description of Quantum Mechanics and General Relativity, the two theories that respectively govern quantum and gravitational phenomena (see, e.g., [1]).

Quantum Mechanics reigns supremely in low energy ( $E \ll E_p$ ) processes where gravity is negligible. In particular, Quantum Field Theory, following the unification of Special Relativity with Quantum Mechanics, successfully describes all experimental data up to energies currently achievable in the laboratory which are in the TeV range. Several characteristic predictions of the Standard Model of strong, electromagnetic and weak interactions have been very successful as in the case of the discovery of the  $W$  and  $Z$  gauge bosons.

On the other hand, Einstein’s General Relativity successfully describes the motion of macroscopic bodies where quantum effects are negligible.

However, a unified description of these two theories is necessary in order to produce predictions for some interesting situations in which both are required, for example the “Big Bang” - the first moments of the Universe, when gravitational interactions were very strong and the scales involved were all microscopic.

If one simply attempts to quantize General Relativity, in the same sense that Quantum Electro Dynamics is a quantization of Maxwell’s theory, the result is an inconsistent theory. This is due to the fact that Newton’s constant is dimensionful and consequently, the divergences can not be disposed of by the technique of renormalization. In addition to this “renormalizability” problem,

great difficulties of a unified description of Quantum Mechanics and General Relativity originate from their deep incompatibilities. One of the most evident aspects of this incompatibility regards the way in which the geometry of space and time is treated. In the Quantum Mechanics picture spacetime is a fixed arena where quantum observables (such as position of a particle) are described. But in General Relativity spacetime can not be treated as a fixed background since it acquires a geometrodynamical structure.

The lack of reliable data on the spacetime at very small distance scales (i.e. for very high energy particles) has led to the proposal of various models for Quantum Gravity, in particular, “String Theory” [2, 3] and “Loop Quantum Gravity” [4, 5, 6, 7]. These models are sometimes very different in the way they approach the technical and conceptual problems emerging from a Quantum-Gravity theory; however, they lead to a common Quantum-Gravity intuition: from any approach to the unification of General Relativity and Quantum Mechanics emerges the idea of a limitation to the localization of the spacetime point. Different arguments can be produced to identify the Planck scale, here intended as the length scale  $L_p = \sqrt{\frac{\hbar G}{c^3}} \simeq 1.6 \cdot 10^{-35} m$ , the inverse of the Planck (energy) scale  $E_p$ , as the special scale at which quantum and gravitational effect are equally important and the description of spacetime so far adopted has to be radically reviewed to accommodate the limitation on localization [8, 9, 10, 11, 12]. For example, in the case of spatial interval one would expect an uncertainty principle of the type

$$\delta x \gtrsim L_p.$$

Intuitively, it is easy to realize how such a limitation could derive from considerations of Quantum Mechanics and General Relativity. From basilar equations of Quantum Mechanics follow that to have a good resolution on small distances it is necessary to use probe particles of high energy to do the measure. But a very energetic particle generates an intense gravitational field that modifies the metric, introducing so a new source of uncertainty on the measure. Thereby, increasing the probes energy one reduces one of the contributions to the total measure uncertainty, but inevitably increases other contributions. Another way to reach the same conclusion can be based on the observation that no particle can be localized in a region of linear dimensions inferior to its own Schwarzschild radius, since the event horizon would then interfere with the use of a probe. The “Schwarzschild-radius uncertainty” in localization, which increases with the particle mass, for a particle of mass of order  $L_p^{-1}$  leads to a localization uncertainty which is also of order  $L_p$ . This is due to the fact that if we combine this gravity-induced “Schwarzschild-radius uncertainty”  $\delta x \geq r_g \sim GM$ , where  $G = L_p^2$  (in natural units  $c = \hbar = 1$ ), with the well established “Compton-wavelength uncertainty” which decreases with the particle

mass,  $\delta x \geq 1/M$ , for a particle of mass of order  $L_p^{-1}$  one cannot do any better than Planck-length localization.

In this scenario it is conceivable that at Plank-length distance scales geometry can have a form which is quite different from the classical one with which we are familiar at large scales. The description of spacetime as a differentiable manifold might need a revision and a new description of geometry might lead to a development of a completely new understanding of physics.

The formalism of noncommutative geometry, which is adopted by this thesis work, is among the most studied possibilities for such a new description of spacetime structure. It essentially assumes ([13]) that one can describe algebraically Quantum Gravity corrections replacing the traditional (Minkowski) spacetime coordinates  $x_\mu$  with Hermitian operators  $\hat{x}_\mu$  that satisfies commutation relation of the type:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}(\hat{x}) .$$

A noncommutative spacetime of this type embodies an impossibility to fully know the short distance structure of spacetime, in the same way that in the phase space of the ordinary Quantum Mechanics there is a limit on the localization of a particle. This fact agrees with the above mentioned intuition of a limitation to localization in the Quantum-Gravity framework.

There is a wide literature on the simplest, so-called “canonical”, noncommutativity characterized by commutators of the coordinates of the type

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} ,$$

where  $\theta_{\mu\nu}$  is a coordinate-independent matrix of dimensionful parameters.

In this thesis we consider another much studied noncommutative spacetime, the  $\kappa$ -Minkowski spacetime, characterized by the commutation relations:

$$[\hat{x}_j, \hat{x}_0] = i\lambda\hat{x}_j \qquad [\hat{x}_j, \hat{x}_k] = 0 ,$$

where  $\lambda^1$  has the dimensions of a length. This type of noncommutativity is an example of “Lie-Algebra-type” noncommutativity in which commutation relations among spacetime coordinates exhibit a linear dependence on the spacetime coordinate themselves.

Recently  $\kappa$ -Minkowski gained remarkable attention due to the fact that it provides an example of noncommutative spacetime in which Lorentz symmetries are preserved as deformed (quantum) symmetries. The quantum deformation and even a break down of Lorentz symmetry is not surprising for a

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<sup>1</sup>Rather than the length scale  $\lambda$  a majority of authors use the energy scale  $\kappa$ , which is the inverse of  $\lambda$  ( $\lambda \rightarrow \frac{1}{\kappa}$ ).

quantum spacetime because of the existence of a minimum spatial length that is not a Lorentz invariant concept. If ordinary Lorentz invariance was preserved, we could always perform a boost and squeeze any given length as much as we want and therefore a minimal length could not exist. If a minimal length really exists we have to contemplate the possibility that the Lorentz invariance is lost. The peculiarity of  $\kappa$ -Minkowski spacetime is that the symmetry is lost as classical symmetry but preserved as “quantum symmetry” in a sense which will be discussed in detail in this work.

The fact that symmetries are deformed in  $\kappa$ -Minkowski has emerged in [14, 15] where  $\kappa$ -Minkowski has been connected with a dimensionful deformation of the Poincaré algebra called  $\kappa$ -Poincaré.

The analysis of the physical implications of the deformed  $\kappa$ -Poincaré algebra have led to interesting hypotheses about the possibility that in  $\kappa$ -Minkowski particles are submitted to modified dispersion relations [16]. Since the growing sensitivity and accuracy of the astrophysical observations renders experimentally accessible such modified dispersion relations (see, e.g., [24] and [26]), there is now strong interest on a systematic analysis of a field theory in  $\kappa$ -Minkowski.

In this work we want to investigate the symmetries of  $\kappa$ -Minkowski non-commutative spacetime connected with the translations sector of  $\kappa$ -Poincaré algebra for a free scalar field. Symmetries are introduced directly at the level of the action, following very strictly commutative field theory in which the symmetry of a theory is defined as transformation of coordinates that leaves invariant the action of the theory. Our analysis, in complete analogy with [17], will be based on the generalization of the Noether theorem within the most studied theory [18], [19], [20] formulated in  $\kappa$ -Minkowski spacetime for a scalar field  $\Phi(x)$  governed by the Klein-Gordon-like equation

$$C_\lambda(P_\mu)\Phi = \left[ \left( \frac{2}{\lambda} \sinh \frac{\lambda}{2} P_0 \right)^2 - e^{\lambda P_0} \vec{P}^2 \right] \Phi = m^2 \Phi.$$

In [17] has been showed that the previous failure to derive energy-momentum conserved charges associated with the  $\kappa$ -Poincaré translation transformations were due to the adoption of a rather naive description of translation transformations, which in particular did not take into account the properties of the noncommutative  $\kappa$ -Minkowski differential calculus. By taking into account the properties of the differential calculus one encounters no obstruction in following all the steps of Noether analysis and obtain an explicit formula relating fields and energy-momentum charges. [17] used the invariance of the theory under the four  $\kappa$ -Poincaré translation transformations and a four-dimensional translational invariant calculus proposed by Majid and Oeckl [21] to find four energy-momentum conserved charges, showing that Hopf algebra can be used to describe genuine spacetime symmetries.

The choice of the vector fields generalizing the notion of derivative in  $\kappa$ -Minkowski represents a key point of our line of analysis. In fact, in the commutative case there is only one (natural) differential calculus involving the conventional derivatives, whereas in the  $\kappa$ -Minkowski case (and in general in a noncommutative spacetime) the introduction of a differential calculus is a more complex problem and, in particular, it is not unique. In our analysis we focus on a possible choice, different from [17], of differential calculus in  $\kappa$ -Minkowski: the “five-dimensional bicovariant calculus” introduced in [22]. The vector fields corresponding to this differential calculus have in fact special covariance properties: they transform under  $\kappa$ -Poincaré in the same way that the ordinary derivatives (i.e. the vector fields associated to the differential calculus in the commutative Minkowski space) transform under Poincaré. Besides, this is the differential calculus under which the action of the  $\kappa$ -Poincaré group becomes linear.

In Chapter 1 we introduce the Hopf-algebras structures which play a fundamental role in the description of  $\kappa$ -Minkowski noncommutative spacetime and its quantum  $\kappa$ -Poincaré symmetry group. In analogy with canonical noncommutative spacetime, where it is used to introduce fields through the Weyl map [23], we introduce a field in  $\kappa$ -Minkowski through a generalized Weyl map based on the notion of generalized Weyl system. The Weyl-system description allows to introduce a field in  $\kappa$ -Minkowski as a generalized Fourier transform that establishes a correspondence between noncommutative positions coordinates (noncommutative coordinate generators of  $\kappa$ -Minkowski) and some commutative Fourier parameters. Thus, such a generalized transform allows us to rewrite structures living on noncommutative spacetime as structures living on a classical (commutative) but non-Abelian “energy-momentum” space.

However, the interpretation that the Quantum Group language gives to “momenta” as generators of translations (i.e. the real physical particle momenta) is based on the notion of quantum group symmetry. It is puzzling in fact that in the Quantum Group literature it is stated (see, e.g., [19]) that the symmetries of  $\kappa$ -Minkowski can be described by any one of a large number of  $\kappa$ -Poincaré basis of generators. The nature of this claimed symmetry-description degeneracy remains obscure from a physics perspective, in particular we are used to associate energy-momentum with the translation generators and it is not conceivable that a given operative definition of energy-momentum could be equivalently described in terms of different translation generators. The difference would be easily established by testing, for example, the different dispersion relations that the different momenta satisfy (a meaningful physical property, which could, in particular, have observable consequences in astrophysics [24], [25] and cosmology [26]).

In Chapter 2 we present the Noether analysis of translation symmetry in  $\kappa$ -Minkowski with a four-dimensional differential calculus and the four translation generators  $P_\mu$  of the Majid-Ruegg  $\kappa$ -Poincaré basis in the definition of



exterior derivative operator  $d$ , going over the steps of the analysis reported in [17] where the conserved charges associated with the translation sector of the  $\kappa$ -Poincaré symmetry transformation have been obtained. This result confirms that in  $\kappa$ -Minkowski there is a non-linear Planck-scale modification of the energy-momentum relation, but the nonlinearity intervenes in a way that differs significantly from what had been conjectured on the basis of some heuristic arguments.

In Chapter 3 we perform all the steps of the Noether analysis for a free scalar field on  $\kappa$ -Minkowski. In order to have all the instruments for our Noether analysis of  $\kappa$ -Poincaré translations on  $\kappa$ -Minkowski, we present first the five-dimensional differential calculus, introduced by Sitarz [22]. We write the exterior derivative operator  $d$  of a generic  $\kappa$ -Minkowski element and the commutation relations between the one-form generators  $d\hat{x}^A$  and the  $\kappa$ -Minkowski generators  $\hat{x}^\mu$ . Noether analysis requires that the exterior derivative operator  $d$ , defined in terms of five translation generators  $\hat{P}_A$ , functions of the four generators  $P_\mu$ , satisfies the Leibnitz rule and we show that this is in fact what happens.

Once we have introduced all these needed tools we proceed with our Noether analysis, relying on direct explicit manipulations of noncommutative fields, and we investigate explicitly the properties of the 5 “would-be currents” that one naturally ends up considering when working with the 5D differential calculus. To obtain conserved charges we perform 3D spatial integration of the currents and, showing how time derivatives are to be formulated in the 5D-calculus setup, we obtain 5 time-independent charges. In fact, we find that within the 5D-calculus setup some subtleties must be handled when trying to establish the time independence of a noncommutative field and our Noether analysis constructively leads us to identify the proper time derivative operator in  $\kappa$ -Minkowski noncommutative spacetime and to a “conservation equation” for the currents. This will motivate a change of basis for the 5D differential calculus with the introduction of a parameter that can be meaningfully described as time-translation parameter. The rotation of the transformation parameters basis does not affect the Noether analysis in any harmful way and leads to a conserved charge associated with the new time-translation parameter which is a plausible candidate for the energy observable.

The primary objective of this thesis work is an investigation of the role that the five-dimensional differential calculus could have in the description of  $\kappa$ -Minkowski spacetime symmetries in alternative to the four-dimensional differential calculus adopted in [17]. The results of Chapter 3 provides support to the idea that the five-dimensional differential calculus can be used for a Noether analysis of the translation sector of  $\kappa$ -Poincaré and all the worries about the presence of five currents, which produces five charges at the end of the analysis, vanish. Besides, the fact that the 5D differential calculus is bico-variant under the action of the full  $\kappa$ -Poincaré algebra and the basis generators

$\hat{P}_A$  of translations transform under  $\kappa$ -Poincaré action in the same way as the operators  $P_\mu$  in the commutative case transform under the standard Poincaré action, might induce to expect that this analysis leads to classical results (see, e.g., [27]). The analysis of Chapter 3 shows how the linearity of  $\kappa$ -Poincaré action on the commutation relation of the 5D differential calculus induces a highly non-trivial structure of the coalgebra sector of the generators  $\hat{P}_A$  and thereby a non-trivial modification of the quantum symmetry. Recovering the classical results at the end of the analysis would seem less likely than expected and the form of the charges obtained shows that this indeed does not happen.

In Chapter 4 we investigate the possibility to derive an energy-momentum (dispersion) relation involving a plausible candidate for the energy observable, by evaluating the charges obtained in Chapter 3 for some trivial solution of the equation of motion. We see that the dispersion relation in the massless case is classical, as expected by [27, 28], while in the massive case there is a Plank-scale modification leading to a non special-relativistic dispersion relation, differently from [27, 28] prediction. However, it is interesting to notice that this modification vanishes if one increases arbitrary the intensity of the fields, i.e. scaling the classical fields by a factor  $A$ , in the limit  $A \rightarrow \infty$ , the special-relativistic relation is reestablished with a mass  $m^R = A^2 m$ .



# Chapter 1

## Noncommutative Geometry and $\kappa$ -Minkowski spacetime

In the first part of this chapter we give a brief overview of Noncommutative Geometry and we introduce the Hopf-algebras structures which play a fundamental role in the description of  $\kappa$ -Minkowski spacetime and its quantum  $\kappa$ -Poincaré symmetry group. In the second part we analyze the symmetries of the deformed Poincaré group on  $\kappa$ -Minkowski and we introduce fields through the powerful concept of Weyl maps. Thus, at the end of the chapter we will be able to write a mass Casimir and a deformed Klein-Gordon equation for a free scalar field in  $\kappa$ -Minkowski.

### 1.1 Preliminaries on Noncommutative Geometry

The Quantum Mechanics phase space, i.e. the space of microscopic states of a quantum particle, provides the first example of noncommutative space. It is defined replacing canonical variables of position and momentum of a particle  $(q_j, p_j)$  with self-adjoint operators  $(\hat{q}_j, \hat{p}_j)$  satisfying Heisenberg's commutation relations

$$[\hat{q}_j, \hat{p}_k] = i\hbar\delta_{jk}, \quad j, k = 1, 2, 3 \quad (1.1)$$

from which follows the Heisenberg uncertainty principle

$$\delta\hat{q}_j\delta\hat{p}_k \geq \frac{\hbar\delta_{jk}}{2}. \quad (1.2)$$

This principle establishes the existence of an accuracy limitation for the measurement of the coordinates and the corresponding momenta of a particle.

Consequently, the quantization of phase space can be viewed as the smearing out of a classical manifold, replacing the notion of a point with that of a Planck cell. The idealized classical situation in which one can simultaneously determine the exact position-momentum measurements is obtained in the limit  $\hbar \rightarrow 0$ , where the phase space becomes a continuum manifold.

A very similar idea led to apply noncommutativity to spacetime itself. The idea of a new structure of spacetime came in the late 40's from Snyder [29] in order to solve the short-distance singularities of the quantum field theory. Later on, the attention was focused on a general noncommutative spacetime of Lie-algebra type with central extension, characterized by the commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} + i\zeta_{\mu\nu}^\alpha \hat{x}_\alpha \quad (1.3)$$

with coordinate-independent  $\theta_{\mu\nu}$  and  $\zeta_{\mu\nu}^\alpha$ ; in particular, the attention was concentrated on the canonical noncommutative spacetime, characterized simply by Heisenberg-like commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}. \quad (1.4)$$

As in case of quantum phase space, this spacetime prescription can be viewed as the smearing out of the classical manifold losing the notion of the *point*: in fact, a Heisenberg-type uncertainty principle implies that the notion of the point is replaced by an analogous of the Planck cell of the quantum phase space.

In the literature there exist principally two main approaches to Noncommutative Geometry. In this work we are interested in how it emerges in the Quantum Groups framework<sup>1</sup> that, for the applications to Quantum Gravity, reflects more the intuition on the meaning of the Planck length  $L_p$  as the length parameter in which the localization indetermination of the spacetime points is manifest, due to the noncommutativity of coordinates. From this point of view it was Woronowicz [31] who initiated a systematic study of the “noncommutative differential geometry” built on some “pseudogroups” that are the generalization of the standard Lie groups related to the commutative differential

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<sup>1</sup>In the other widely spread approach, largely due to Connes [30], in order to be able to construct field theories on noncommutative spaces in the same way as on traditional commutative spaces the attention is no more focused on the spacetime but on the algebra of functions. In particular, Connes' idea of Noncommutative Geometry is based on the re-formulation of the manifold geometry in terms of C\*-algebras of functions defined over the manifold, with a generalization of the corresponding results of differential geometry to the case of a noncommutative algebra of functions. The characterization of the Hilbert space and the Dirac operator become the main ingredients and there is no more an explicit reference neither to the spacetime or the coordinates, which in general can also do not exist in a concrete form.

geometry.

The study of these “quantum groups” algebras have a fundamental role in the description of symmetries of noncommutative spacetime. Our attention in this work is on  $\kappa$ -Minkowski spacetime, characterized by the commutation relations:

$$[\hat{x}_j, \hat{x}_0] = i\lambda\hat{x}_j \qquad [\hat{x}_j, \hat{x}_k] = 0. \qquad (1.5)$$

On the one hand, this example of Lie-algebra-type noncommutativity which introduces a length deformation parameter ( $\lambda$ ) appears as a natural candidate for a quantized spacetime with a new limitation on the measurability of geometrical quantity. In fact the results of [13], [32] show how  $\kappa$ -Minkowski provides a particular realization of the minimal length concept.

On the other hand, we will show in section (1.3) how  $\kappa$ -Minkowski can be seen as the dual Hopf algebra (the concept of duality will be explained in detail in (1.2.1) and Appendix A where Bicrossproduct Hopf algebras are introduced in detail) of the momentum sector of the  $\kappa$ -Poincaré algebra.

Thus, for a clearer comprehension of the framework we are working in, we give in section (1.2) a brief description of the Quantum groups language in order to use it in the rest of the chapter to introduce  $\kappa$ -Poincaré and  $\kappa$ -Minkowski and investigate their mathematical structure and symmetries.

## 1.2 Quantum Groups and their emergence in Noncommutative Geometry

Quantum Groups or Hopf algebras are a generalization of the ordinary groups (i.e. collections of transformations on a space that are invertible). They have a rich mathematical structure and numerous roles in physical situations where ordinary groups are not adequate. Quantum Groups allows us to generalize many “classical” physical ideas in a completely self-consistent way. This generalization is realized through a “deformation” induced by the presence of one or more parameters. The classical case is recovered by setting these parameters to some fixed values. A very similar case of quantization is represented by Quantum Mechanics, in which the deformation is introduced by the Planck constant  $\hbar$ , and the classical case is recovered in the limit  $\hbar \rightarrow 0$ . As we will show below, Quantum Groups have structure, such as the coproduct and the antipode, that generalize some properties of ordinary groups, such as the representation on a vector-space tensor product or the existence of an inverse. This properties are at the basis of the applications of Quantum Groups in a wide physical domain, from Statistical Physics to Quantum Gravity.

The mathematical structure of  $\kappa$ -Minkowski has emerged and has been described in the context of studies which relate noncommutative spaces and the world of quantum groups. Just like Lie groups and the homogeneous spaces associated to them provide a complete description of the classical differential geometry, so quantum groups and the homogeneous quantum spaces associated provide a wide class of examples on which it is possible to build and develop a noncommutative geometry.

In particular, among the several classes of quantum groups and algebras, we can consider  $\kappa$ -Poincaré, built as a deformed algebra of the usual relativistic symmetries.

A special class of Quantum Groups (called of “bicrossproduct” type) was largely investigated by S. Majid in the approach to Planck-scale Physics [33]. As we will see below, this line of research represents an important point for our study of  $\kappa$ -Minkowski. In fact, as shown in section (1.3), for a particular choice of the generators basis,  $\kappa$ -Poincaré algebra has a manifest structure of bicrossproduct Hopf algebra with the properties of duality that enable to identify  $\kappa$ -Minkowski as the space where  $\kappa$ -Poincaré acts on in a covariant way, i.e the commutation relations that characterize  $\kappa$ -Minkowski remain unchanged under the action of  $\kappa$ -Poincaré algebra (in Appendix A we provide the demonstration of  $\kappa$ -Poincaré invariant action on  $\kappa$ -Minkowski).

The nature of quantum groups will be clearer at the end of this section, but we can anticipate here the fact that a quantum group is a noncommutative noncocommutative Hopf algebra. Let us clarify the term “quantum group”. The term “group” refers to the correspondence between topological groups and commutative Hopf algebras, since it is always possible to associate a commutative Hopf algebra to every topological compact group  $G$  and all the properties of the group  $G$  can be reformulated in terms of the Hopf algebra  $A = C(G)$ , the space of continuum functions on  $G$ . The term “quantum” refers to the deformation of the Hopf algebra  $A$  into a certain noncommutative Hopf algebra  $A_q$ , where  $q$  is the deformation parameter. In practice we do not deform the group  $G$ , but its dual object  $A = C(G)$ . Thereby the quantum groups category can be considered dual to that of noncommutative noncocommutative Hopf algebras. In other words a quantum group can be considered as the geometric object, with noncommutative coordinates, corresponding to a general Hopf algebra. As we will see, this particular mathematical object has the property that its dual space turns out to be again a Hopf algebra. In particular,  $\kappa$ -Minkowski itself will be a Hopf algebra with a dual space of momenta that will have the typical commutative structure and where we will define our field theory.

In order to introduce the notion of quantum group symmetry that should preserve a covariant action over the associated homogeneous space, we provide in this section some notions about the definition of Hopf Algebra (or Quantum Group), which is relevant for our description of quantum deformations of Poincaré group.

### 1.2.1 Hopf Algebras

The central structure of all quantum groups theory is that of Hopf algebra. The extra structures that characterize a Hopf Algebra with respect to a Lie algebra turn out to be very useful in order to translate in the mathematical language some physical properties. In particular, one finds that it is necessary to introduce some new mathematical in the rules of composition of representations. Let us start with the definition of  $\mathbb{C}$ -algebra (associative algebra with unity).

**Definition.** A vector space  $A$  on the complex field  $\mathbb{C}$  endowed with two maps  $m$  ( $m : A \otimes A \rightarrow A$ ) and  $\eta$  ( $\eta : \mathbb{C} \rightarrow A$ ) is defined  $\mathbb{C}$ -algebra if  $m$  and  $\eta$  satisfy:

$$m(m \otimes 1) = m(1 \otimes m) \quad (\text{associativity}) \quad (1.6)$$

$$m(1 \otimes \eta) = m(\eta \otimes 1) = id \quad (\text{unity}) \quad (1.7)$$

where  $id : A \rightarrow A$  denotes the identity map on  $A$ .

A representation of an algebra  $A$  over a vector space  $V$  is a set  $(V, \rho)$ , where  $\rho$  is a linear map from  $A$  to the space of linear operator in  $V$ ,  $Lin(V)$ , satisfying

$$\rho(ab) = \rho(a)\rho(b) \quad a, b \in A.$$

If we now take two vector spaces  $V_1$  and  $V_2$  and we want to use the representations of the algebra  $A$  on them,  $(V_1, \rho)$  and  $(V_2, \rho)$ , to determine the representation of  $A$  on the tensor product of the spaces  $(V_1 \otimes V_2, \rho)$  we need a new structure in order to satisfy linearity and homomorphism property, and to reflect the associativity of the algebra. This structure is the coproduct, defined as a linear map that splits an algebra element into a sum of elements belonging to the tensor product of algebras:

$$\Delta : A \rightarrow A \otimes A. \quad (1.8)$$

In this way the coproduct is a sum of tensor products and is indicated as  $\Delta(a) = \sum_i a_{(1)}^i \otimes a_{(2)}^i$  or, in the Sweedler notation,  $\Delta(a) = a_{(1)} \otimes a_{(2)}$ .

Using the coproduct the representation of  $A$  is given by

$$\rho(a) = ((\rho_1 \otimes \rho_2) \cdot \Delta(a))(v_1 \otimes v_2) \quad a \in A. \quad (1.9)$$

To ensure the homomorphism property of  $\Delta$  and associativity of the algebra,



$\Delta$  must satisfy these conditions

$$\Delta(ab) = \Delta(a)\Delta(b), \quad (1.10)$$

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta \quad (coassociativity). \quad (1.11)$$

It is then natural to generalize also the unity in the so-called co-unity, a map  $\epsilon$  such that:

$$\epsilon : \quad A \rightarrow \mathbb{C} \quad (1.12)$$

$$(1 \otimes \epsilon) \cdot \Delta = (\epsilon \otimes 1) \cdot \Delta = id \quad (counity). \quad (1.13)$$

In this way we can give the definition of a coalgebra. A *coalgebra*  $C$  is a vector space over a field  $\mathbb{C}$  endowed with a linear coproduct  $\Delta : C \rightarrow C \otimes C$  and a linear counit  $\epsilon : C \rightarrow \mathbb{C}$ , which satisfies the *coassociativity* (1.11) and *counity* (1.13) properties.

**Definition.** A *Hopf algebra*  $(H, m, \eta; \Delta, \epsilon, S)$  is a vector space that is both an algebra and a coalgebra in a compatible way endowed with a linear antipode map  $S : H \rightarrow H$  such that:

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \eta\epsilon; \quad (1.14)$$

the compatibility is given by the following homomorphism properties

$$\Delta(ab) = \Delta(a)\Delta(b) \quad \Delta(1) = 1 \otimes 1 \quad (1.15)$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b) \quad \epsilon(1) = 1 \quad (1.16)$$

for all  $a, b \in H$ . By the definition (1.14) it follows that the antipode is unique and satisfies:

$$S(a \cdot b) = S(a)S(b), \quad S(1) = 1 \quad (algebra \ antirepresentation) \quad (1.17)$$

$$(S \otimes S)\Delta(a) = \tau\Delta S(a) \quad (1.18)$$

$a, b \in H$  and  $\tau$  represent the flip map  $\tau : \tau(a \otimes b) = b \otimes a$ . In a Hopf algebra the antipode plays a role that generalizes the concept of group inversion.

We want to introduce now the notion of *duality*. Two Hopf algebras  $H$  and  $H^*$  are said to be *dually paired* if there exists a non degenerate inner product  $\langle, \rangle$  such that the following axioms are satisfied

$$\langle ab, c \rangle = \langle a \otimes b, \Delta(c) \rangle \quad (1.19)$$

$$\langle 1_{H^*}, c \rangle = \epsilon(c) \quad (1.20)$$

$$\langle \Delta(a), c \otimes d \rangle = \langle a, cd \rangle \quad (1.21)$$

$$\epsilon(a) = \langle a, 1_H \rangle \quad (1.22)$$

$$\langle S(a), c \rangle = \langle a, S(c) \rangle \quad (1.23)$$

where  $a, b \in H$  and  $\langle a \otimes b, c \otimes d \rangle = \langle a, c \rangle \langle b, d \rangle$ .

Note that the relations above may be used constructively, i.e. given a Hopf algebra  $H$ , one can construct a dually paired Hopf algebra  $H^*$ ; this method is used to construct the spacetime coordinate algebra from the Hopf algebra of the translation generators, as we will show in the following for  $\kappa$ -Minkowski spacetime, obtained by duality from the momenta sector of the  $\kappa$ -Poincaré Hopf algebra.

One can show that to each proposition over an algebra corresponds a dual proposition over the dual structure that is obtained by substituting each operation over the algebra with the corresponding operation over the dual structure. In this way one can establish, for example, that the dual of a *commutative* Hopf algebra is *co-commutative*, and vice-versa. In fact, from the commutativity of  $H$  ( $cd = dc \ \forall c, d \in H$ ) it follows

$$\begin{aligned} \langle a_{(1)}, c \rangle \langle a_{(2)}, d \rangle &= \langle \Delta(a), c \otimes d \rangle = \langle a, cd \rangle = \langle a, dc \rangle = \\ &= \langle a_{(1)}, d \rangle \langle a_{(2)}, c \rangle = \langle a_{(2)}, c \rangle \langle a_{(1)}, d \rangle, \end{aligned}$$

comparing the first and the last members we find that  $\tau\Delta = \Delta$ .

### 1.3 Deformation of the Poincaré algebra and $\kappa$ -Minkowski spacetime

In the framework of Quantum Groups, the deformation of the Poincaré group has attracted much attention in the early 1990s mostly for the motivation arising from Quantum Gravity, in which a loss of the classical Lorentz symmetry would not be surprising due to the existence of a *minimum length*. Different approaches have been attempted in this direction, but interesting developments have been found in looking for a deformation of the algebra rather than the group. Following the very powerful technique of contraction procedure introduced in [34], which consider the q-deformation of the anti-De Sitter algebra  $SU(2)_Q$ , one recovers a quantum deformation  $U_\kappa(P_4)$  of the Poincaré algebra  $P_4$  which depends on a *dimensionful* parameter  $\kappa$ . In this way a fundamental length  $\lambda = \kappa^{-1}$  enters the theory. This quantum algebra has been obtained firstly in [35] in the so-called *standard basis*, whose characteristic commutation relations are:

$$\begin{aligned}
[P_\mu, P_\nu] &= 0, \\
[M_j, P_0] &= 0, \quad [M_j, P_k] = i\epsilon_{jkl}P_l, \\
[N_j, P_0] &= iP_j, \quad [N_j, P_k] = i\delta_{jk}\lambda^{-1}\sinh\lambda P_0, \\
[M_j, M_k] &= i\epsilon_{jkl}M_l, \quad [M_j, N_k] = i\epsilon_{jkl}N_l, \\
[N_j, N_k] &= -i\epsilon_{jkl}(M_l \cosh\lambda P_0 - \frac{\lambda^2}{4}P_l \vec{P} \cdot \vec{M}), \tag{1.24}
\end{aligned}$$

where  $P_\mu$  are the four-momentum generators,  $M_j$  are the spatial rotation generators and  $N_j$  are the boost generators. The algebra obtained in this way contains the subalgebra of the classical rotations  $O(3)$ . The cross-relations between the boost and the rotation generators are instead deformed, and consequently the full Lorentz sector do not form a sub-algebra. The coalgebra

sector of the  $\kappa$ -Poincaré standard basis is given by:

$$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0, \quad \Delta(P_j) = P_j \otimes e^{\frac{\lambda P_0}{2}} + e^{-\frac{\lambda P_0}{2}} \otimes P_j, \quad (1.25)$$

$$\Delta(M_j) = M_j \otimes 1 + 1 \otimes M_j, \quad (1.26)$$

$$\Delta(N_j) = N_j \otimes e^{\frac{\lambda P_0}{2}} + e^{-\frac{\lambda P_0}{2}} \otimes N_j + \frac{\lambda}{2} \varepsilon_{jkl} (P_k \otimes M_l e^{\frac{\lambda P_0}{2}} + e^{-\frac{\lambda P_0}{2}} M_k \otimes P_l). \quad (1.27)$$

The mass Casimir  $C_\lambda(P)$ , i.e. the function that commutes with all the generators of the algebra, is:

$$C_\lambda(P) = \left( \frac{2}{\lambda} \sinh \frac{\lambda P_0}{2} \right)^2 - \vec{P}^2 \xrightarrow{\lambda \rightarrow 0} P_0^2 - \vec{P}^2; \quad (1.28)$$

it provides a deformation of the Casimir of the Poincaré algebra  $C(P) = P_0^2 - \vec{P}^2$ .

In the quantum Groups language it is said that the pair of a Hopf algebra and its dual determines a generalized phase space, i.e. the space of the generalized momenta and the corresponding generalized coordinates. The quantum algebra  $U_k(P_4)$  contains a translation subalgebra, and it is natural to consider the dual of the enveloping algebra of translations as a  $\kappa$ -Minkowski space. This space must necessarily be noncommutative, because the duality axioms (see 1.19) state that a non-cocommutative algebra in the momenta corresponds to a noncommutative algebra in the spacetime coordinates. So, the non-cocommutative relations (1.25) imply that the generators of the dual space (spacetime coordinates) do not commute.

However, one expects that the quantum deformation of a group symmetry (such as  $U_k(P_4)$ ) represents, in some sense, a “quantum symmetry” for the corresponding homogenous space. In our case, for example,  $\kappa$ -Poincaré is expected to act on  $\kappa$ -Minkowski spacetime in a covariant way, preserving its algebra structure. For this reason, a new  $\kappa$ -Poincaré basis has been introduced, in which the “covariance” of its action on  $\kappa$ -Minkowski is clearly manifest. This is the case of the Majid-Ruegg *bicrossproduct basis* introduced in [15].

One has a large freedom in the choice of the generators of the quantum algebra  $U_k(P_4)$ . One can define a very large number of basis through nonlinear combinations of the generators. Thereby the choice of the generators of  $U_k(P_4)$  is not unique: different choices of the basis generators modify the form of the  $\kappa$ -Poincaré Hopf algebra in the algebra sector (i.e. the commutation relations among generators) and in the coalgebra sector (i.e. the form of the coproduct and the counit). It has been found in [15] that the  $\kappa$ -deformed Poincaré

algebra, in a particular choice of generators basis, has a manifest structure if bicrossproduct Hopf algebra  $U(so(1, 3)) \bowtie \triangleleft T$  (see Appendix A), i.e. the semidirect product of the classical Lorentz group  $so(1, 3)$  acting in a deformed way on the momentum sector  $T$ , and in which also the coalgebra is semidirect with a back-reaction of the momentum sector on the Lorentz rotations. The following change of variables:

$$\mathcal{P}_0 = -P_0, \quad \mathcal{P}_j = -P_j e^{\frac{\lambda P_0}{2}}, \quad \mathcal{N}_j = N_j e^{\frac{\lambda P_0}{2}} - \frac{\lambda}{2} \epsilon_{jkl} M_k P_l e^{\frac{\lambda P_0}{2}} \quad (1.29)$$

leads to the  $\kappa$ -Poincaré algebra in the so-called *Majid-Ruegg bicrossproduct basis* in which the Lorentz sector is not deformed. The deformation occurs only in the cross-relations between the Lorentz and translational sectors

$$[\mathcal{P}_\mu, \mathcal{P}_\nu] = 0$$

$$[M_j, \mathcal{P}_0] = 0$$

$$[M_j, \mathcal{P}_k] = i\epsilon_{jkl} \mathcal{P}_l$$

$$[\mathcal{N}_j, \mathcal{P}_0] = i\mathcal{P}_j$$

$$[\mathcal{N}_j, \mathcal{P}_k] = i\delta_{jk} \left( \frac{1}{2\lambda} (1 - e^{-2\lambda \mathcal{P}_0}) + \frac{\lambda}{2} \mathcal{P}^2 \right) - i\lambda \mathcal{P}_j \mathcal{P}_k \quad (1.30)$$

and the Lorentz subalgebra remains classical

$$[M_j, M_k] = i\epsilon_{jkl} M_l$$

$$[M_j, \mathcal{N}_k] = i\epsilon_{jkl} \mathcal{N}_l$$

$$[\mathcal{N}_j, \mathcal{N}_k] = -i\epsilon_{jkl} M_l. \quad (1.31)$$

The coproducts are given by

$$\Delta(\mathcal{P}_0) = \mathcal{P}_0 \otimes 1 + 1 \otimes \mathcal{P}_0$$

$$\Delta(\mathcal{P}_j) = \mathcal{P}_j \otimes e^{-\lambda \mathcal{P}_0} + 1 \otimes \mathcal{P}_j$$

$$\Delta(M_j) = M_j \otimes 1 + 1 \otimes M_j$$

$$\Delta(\mathcal{N}_j) = \mathcal{N}_j \otimes e^{-\lambda \mathcal{P}_0} + 1 \otimes \mathcal{N}_j - \lambda \epsilon_{jkl} \mathcal{M}_k \otimes P_l \quad (1.32)$$

and the antipodes are

$$S(\mathcal{P}_j) = -\mathcal{P}_j e^{\lambda \mathcal{P}_0}, \quad S(\mathcal{P}_0) = -\mathcal{P}_0, \quad S(M_j) = -M_j,$$

$$S(\mathcal{N}_j) = -\mathcal{N}_j e^{\lambda \mathcal{P}_0} - \lambda \epsilon_{jkl} M_k \mathcal{P}_l e^{\lambda \mathcal{P}_0}. \quad (1.33)$$

The mass Casimir of this algebra, i.e. the function that commute with all the generators of the algebra, is given by:

$$C_\lambda(P) = \left( \frac{2}{\lambda} \sinh \frac{\lambda P_0}{2} \right)^2 - e^{\lambda P_0} \vec{P}^2 \xrightarrow{\lambda \rightarrow 0} P_0^2 - \vec{P}^2. \quad (1.34)$$

This deformation of the Poincaré Casimir has led to many discussion about the phenomenological implications of a deformed group symmetry. This is essentially due to the connections of  $\kappa$ -Poincaré with  $\kappa$ -Minkowski spacetime, in which the relation (1.34) is considered to have the interpretation of deformed dispersion relation for particle.

$\kappa$ -Minkowski noncommutative spacetime, whose coordinates satisfy the commutation relations (1.5), is shown to be the spacetime associated to  $\kappa$ -Poincaré algebra. In fact, expressing the  $\kappa$ -Poincaré generators in this basis (which from now on we denote by  $(P_\mu, M_j, N_j)$ ), it is possible to see that  $\kappa$ -Poincaré acts covariantly as a Hopf algebra on  $\kappa$ -Minkowski spacetime as shown in Appendix A. In this way the commutation relations (1.5) that characterize  $\kappa$ -Minkowski remain unchanged under the action of  $\kappa$ -Poincaré algebra, this is consistent with the notion of quantum group symmetry that should preserve a covariant

action over the associated homogeneous space. In the limit  $\lambda \rightarrow 0$  one recovers the standard Minkowski space, with the ordinary Poincaré group.

In this basis it is very easy to show that the dual Hopf algebra  $T^*$  of the translation sector  $T$  of the  $\kappa$ -Poincaré algebra,  $T \subset U_\kappa(P_4)$ , is the Hopf algebra of the  $\kappa$ -Minkowski generators  $\hat{x}_\mu$ . We can assume the duality relations

$$\langle \hat{x}_\mu, P_\nu \rangle = -i\eta_{\mu\nu} \quad (1.35)$$

and, applying the duality axioms, we can determine the Hopf algebra of  $\hat{x}_\mu$  if we know the Hopf algebra of  $P_\mu$ . For example, using the axiom (1.19) and the coproduct (1.32), one finds:

$$\begin{aligned} \langle [\hat{x}_j, \hat{x}_0], P_k \rangle &= \langle \hat{x}_j \otimes \hat{x}_0, \Delta(P_k) \rangle - \langle \hat{x}_0 \otimes \hat{x}_j, \Delta(P_k) \rangle = \\ &= \langle \hat{x}_j \otimes \hat{x}_0, P_k \otimes e^{-\lambda P_0} + 1 \otimes P_k \rangle + \\ &- \langle \hat{x}_0 \otimes \hat{x}_j, P_k \otimes e^{-\lambda P_0} + 1 \otimes P_k \rangle = \\ &= \langle \hat{x}_j, P_k \rangle \langle \hat{x}_0, e^{-\lambda P_0} \rangle - \langle \hat{x}_0, 1 \rangle \langle \hat{x}_j, P_k \rangle = \\ &= -\lambda \langle \hat{x}_j, P_k \rangle \langle \hat{x}_0, P_0 \rangle = \langle i\lambda \hat{x}_j, P_k \rangle, \end{aligned}$$

from which it follows:

$$[\hat{x}_j, \hat{x}_0] = i\lambda \hat{x}_j;$$

this is the non-zero commutator between space and time “coordinates” of  $\kappa$ -Minkowski.

## 1.4 Fields in $\kappa$ -Minkowski and Weyl maps

For the development of a field theory in  $\kappa$ -Minkowski it is fundamental to have a convenient characterization of the concept of field as function of noncommutative variables. In our description and handling of functions of the noncommuting coordinates (fields in the noncommutative geometry) an important role

will be played by *Weyl maps*, which allow to introduce structures for the functions of the noncommuting coordinates in terms of the corresponding structures that are meaningful for the ordinary functions.

Weyl maps establish a correspondence between elements of  $\kappa$ -Minkowski and analytic functions of four variables  $x_\mu$  that commute. This correspondence is not unique, i.e. there exist several Weyl maps which can be defined. Thereby a coherence criterion for the proposed theories is that of Weyl map choice independence.

Among the several Weyl maps that one can define, it can be useful to consider two explicit examples, which we denote by  $\Omega_R$  and  $\Omega_S$ , so that we get some intuition for the differences which may arise. To characterize the Weyl maps  $\Omega_R$  and  $\Omega_S$  let us consider a simple function  $f(x) = x_j x_0$  of the Minkowski commutative spacetime. The action on the function  $f$  of the *time-to-the-right*  $\Omega_R$  Weyl map and of the *time-symmetrized* Weyl map  $\Omega_S$  are the following:

$$\Omega_R(f) = \hat{x}_j \hat{x}_0 \quad \Omega_S(f) = \frac{1}{2}(\hat{x}_j \hat{x}_0 + \hat{x}_0 \hat{x}_j) .$$

These two maps are related to two possible orderings that one can choose for the noncommutative functions of coordinates in  $\kappa$ -Minkowski spacetime.

It is sufficient to specify the Weyl map on the complex exponentials and extend it to the generic function  $\Omega_{R,S}(f(x))$ , whose Fourier transform is  $\tilde{f}(p) = \frac{1}{(2\pi)^4} \int f(x) e^{-ipx} d^4x$ , by linearity

$$\Omega_{R,S}(f(x)) = \int \tilde{f}(p) \Omega_{R,S}(e^{ipx}) d^4p . \quad (1.36)$$

The  $\Omega_R$  Weyl map is implicitly defined through

$$\Omega_R(e^{ipx}) = e^{i\vec{p}\vec{\hat{x}}} e^{-ip_0 \hat{x}_0} , \quad (1.37)$$

while the alternative  $\Omega_S$  Weyl map is such that

$$\Omega_S(e^{ipx}) = e^{-i\frac{p_0 \hat{x}_0}{2}} e^{i\vec{p}\vec{\hat{x}}} e^{-i\frac{p_0 \hat{x}_0}{2}} , \quad (1.38)$$

where  $p_\mu$  are four real commutative parameters. It is so possible, using the definition of Fourier transform we have in Minkowski commutative spacetime, to define the function of  $\kappa$ -Minkowski ordered through the two maps  $\Omega_R$  and  $\Omega_S$  as the Fourier integrals with the  $\kappa$ -Minkowski exponentials ordered through the two maps.

Notice that it is possible to go from time-to-the-right to time-symmetrized ordering through a transformation of the Fourier parameters



$$\Omega_R(e^{ipx}) = \Omega_S(e^{i\vec{p}e^{\frac{\lambda p_0}{2}}\vec{x} - ip_0x_0}). \quad (1.39)$$

In the development of a field theory, with these fields of noncommuting space-time coordinates, the description of products of fields plays of course a central role. And it is useful to describe the product of two fields  $F$  and  $G$  of noncommuting spacetime coordinates in terms of (correspondingly deformed) rule of product for the commuting fields  $f$  and  $g$  through the Weyl map:

$$F = \Omega(f) \quad G = \Omega(g).$$

Of course, as a result of the noncommutativity, the product  $FG$  cannot be described as  $\Omega(fg)$ . Instead one has that  $FG = \Omega(f \star g)$ , where

$$(f \star g) = \Omega^{-1}(\Omega(f)\Omega(g)) \quad (1.40)$$

is the “ $\star$ -product” (often also called Moyal product).

In the case of the  $\Omega_R$  and  $\Omega_S$  Weyl maps for  $\kappa$ -Minkowski one finds:

$$\Omega_R(e^{ipx}) \cdot \Omega_R(e^{iqx}) = \Omega_R(e^{ipx} \star_R e^{iqx}) = \Omega_R(e^{i(\vec{p} + \vec{q}e^{-\lambda p_0})\vec{x} - i(p_0 + q_0)x_0}),$$

$$\Omega_S(e^{ipx}) \cdot \Omega_S(e^{iqx}) = \Omega_S(e^{ipx} \star_S e^{iqx}) = \Omega_S(e^{i(\vec{p}e^{\frac{\lambda q_0}{2}} + \vec{q}e^{\frac{-\lambda p_0}{2}})\vec{x} - i(p_0 + q_0)x_0}).$$

The Weyl map can also be used to introduce a notion of integration in the noncommutative spacetime. We can assume a rule of integration that is naturally expressed using the  $\Omega_R$  Weyl map

$$\int_R \Omega_R(f) = \int f(x) d^4x \quad (1.41)$$

which states that the integral of a right-ordered function of  $\kappa$ -Minkowski corresponds exactly to the integral of the corresponding commutative function. In this way the right integral of a right-ordered exponential corresponds to the standard delta function:

$$\frac{1}{(2\pi)^4} \int_R e^{i\vec{k}\vec{x}} e^{-ik_0\hat{x}_0} = \frac{1}{(2\pi)^4} \int d^4x \Omega_R^{-1} \Omega_R(e^{ikx}) = \delta^4(k). \quad (1.42)$$

This rule has been largely investigated in literature (see for example [36]). Our

alternative choice of Weyl map would naturally invite us to consider the integration rule

$$\int_S \Omega_S(f) = \int f(x) d^4x. \quad (1.43)$$

Actually these integrals are equivalent, i.e.  $\int_R \Phi = \int_S \Phi$  for each element  $\Phi$  of  $\kappa$ -Minkowski. This is easily verified by expressing the most general element of  $\kappa$ -Minkowski both in its  $\Omega_R$ -inspired form and its  $\Omega_S$ -inspired form

$$\Phi = \int d^4p \tilde{f}(p) \Omega_R(e^{ipx}) = \int d^4p \tilde{f}(p_0, \vec{p} e^{-\frac{\lambda p_0}{2}}) e^{-\frac{3\lambda p_0}{2}} \Omega_S(e^{ipx})$$

and observing that

$$\int_R \Phi = \int_S \Phi = (2\pi)^4 \tilde{f}(0). \quad (1.44)$$

Because of the equivalence we will omit indices  $R$  or  $S$  on the integration symbol.

## 1.5 Free scalar fields in classical Minkowski

While for the canonical noncommutative spacetimes the naive choice of action  $S(\Phi) = \int d^4x \Phi(\partial^2 - M^2)\Phi$  (for free scalar fields) is fully satisfactory, in the description of free scalar fields in  $\kappa$ -Minkowski a nontrivial choice of action emerges very naturally. This originates from the desire to work with a “maximally symmetric” action, and in the case of  $\kappa$ -Minkowski it is possible to introduce an action which is invariant under the 10 Poincaré-like symmetries, but this action has nontrivial form.

In preparation for this  $\kappa$ -Minkowski analysis we find useful to devote this section to a description of the simple action  $S(\Phi) = \int d^4x \Phi(\partial^2 - M^2)\Phi$  for a free scalar field  $\Phi$  in commutative Minkowski spacetime ( $\partial^2 = \partial_\mu \partial^\mu$  is the familiar D’Alembert operator).

Let us start by introducing some notation and convention for the description of symmetry transformations. The most general infinitesimal transformation is of the form  $x'_\mu = x_\mu + \epsilon A_\mu(x)$ , with  $A_\mu$  four real functions of the coordinates.

A field is scalar if  $\Phi'(x') = \Phi(x)$ , and in leading order in  $\epsilon$  one finds

$$\Phi'(x') - \Phi(x) = \partial^\mu \Phi(x) (x_\mu - x'_\mu) = -\epsilon A_\mu(x) \partial^\mu \Phi(x);$$

in terms of the generator  $T$  of the transformation,  $T = iA_\mu(x) \partial^\mu$ , one obtains  $x' = (1 - i\epsilon T)x$  and  $\Phi' = (1 + i\epsilon T)\Phi$ .

Correspondingly the variation of the action can be written as

$$\begin{aligned} S(\Phi') - S(\Phi) &= i\epsilon \int d^4x \left( T\{\Phi(\partial^2 - M^2)\Phi\} + \Phi[\partial^2, T]\Phi \right) = \\ &= i\epsilon \int d^4x \left( TL(x) + \Phi[\partial^2, T]\Phi \right) \end{aligned}$$

and therefore the action is invariant under  $T$ -generated transformations,

$$S(\Phi') - S(\Phi) = 0$$

if and only if

$$\int d^4x \left( TL(x) + \Phi[\partial^2, T]\Phi \right) = 0. \quad (1.45)$$

For the action  $S(\Phi) = \int d^4x \Phi(\partial^2 - M^2)\Phi$  in classical Minkowski spacetime it is well established that the symmetries are described in terms of the classical Poincaré algebra, generated by the elements

$$P_\mu = -i\partial_\mu, \quad M_j = \epsilon_{jkl}x_kP_l, \quad N_j = x_jP_0 - x_0P_j,$$

which satisfy the commutation relations

$$[P_\mu, P_\nu] = 0, \quad [M_j, P_0] = 0, \quad [M_j, P_k] = i\epsilon_{jkl}P_l,$$

$$[M_j, M_k] = i\epsilon_{jkl}M_l, \quad [M_j, N_k] = i\epsilon_{jkl}N_l,$$

$$[N_j, P_0] = iP_j, \quad [N_j, P_k] = i\delta_{jk}P_0, \quad [N_j, N_k] = -i\epsilon_{jkl}M_l. \quad (1.46)$$

The operator  $\partial^2 = -P_\mu P^\mu$  is the first Casimir of the algebra, and of course satisfies  $[\partial^2, T] = 0$ .

For this case of a maximally-symmetric theory in commutative Minkowski spacetime it is conventional to describe the symmetries fully in terms of Poincaré Lie algebra. For  $\kappa$ -Minkowski noncommutative spacetime a description of symmetry in terms of a Hopf algebra turns out to be necessary. But we must stress that essentially the difference between symmetries described in terms of a Lie

algebra and symmetries described in terms of a Hopf algebra resides in the description of the action of symmetry transformations on products of functions: if for all generators  $T_a$  one finds that  $T_a(fg) = [T_a(f)]g + f[T_a(g)]$ , one may say that the coproduct is trivial and a description in terms of a Lie algebra is sufficient, whereas for the case when the coproduct is nontrivial one speaks of a Hopf-algebra symmetry.

Once the algebra properties are specified (action of symmetry transformation on functions of noncommutative coordinates) the property of the counit, coproduct and antipode can always be formally derived, but this will not in general satisfy the Hopf algebra criteria since they may require the introduction of new operators, not included in the algebra sector. If this does not occur (if the counit, coproduct and antipode that one obtains on the basis of the algebra sector can be expressed fully in terms of operators in the algebra) the Hopf-algebra criteria are automatically satisfied.

## 1.6 Symmetry analysis in $\kappa$ -Minkowski spacetime

We want to discuss in this section the form of the action for a free scalar field in  $\kappa$ -Minkowski which most naturally replaces the  $S(\Phi) = \int d^4x \Phi(\partial^2 - M^2)\Phi$  action of the classical-Minkowski case assuming the integration rule (1.44).

A key point is that it is possible to introduce an action for a free scalar field in  $\kappa$ -Minkowski which is invariant under translations, space-rotations and boosts, in the Hopf-algebra sense.

By straightforward generalization of the result (1.45) reviewed in the previous section, a symmetry transformation  $T$  must be such that

$$\int d^4x (T\{\Phi(\partial_\lambda^2 - M^2)\Phi\} + \Phi[\partial_\lambda^2, T]\Phi) = 0, \quad (1.47)$$

if the action takes the form

$$S(\Phi) = \int d^4x \Phi(\partial_\lambda^2 - M^2)\Phi$$

with  $\partial_\lambda^2$  to be determined.

The next step is the description of the Poincaré-like symmetries which will be implemented as invariances of the action. One of course wants to introduce a description of translations, space-rotations and boosts that follows as closely as possible the analogy with the well-established descriptions that apply in the commutative limit  $\lambda \rightarrow 0$ . Since functions in  $\kappa$ -Minkowski can be fully described in terms of Weyl map, and since the Weyl map are fully specified

once given on Fourier exponentials, one can, when convenient, confine the discussion to the Fourier exponentials.

### 1.6.1 Translations

Since in classical Minkowski the translation generator acts according to

$$P_\mu(e^{ikx}) = k_\mu e^{ikx} \quad (1.48)$$

in an analysis of  $\kappa$ -Minkowski based on the time-to-the-right Weyl map it is natural to define translations as generated by the operators  $P_\mu^R$  such that

$$P_\mu^R \Omega_R(e^{ikx}) = k_\mu \Omega_R(e^{ikx}). \quad (1.49)$$

Since, as mentioned, the exponentials  $e^{i\vec{k}\vec{x}}e^{-ik_0\hat{x}_0}$  form a basis of  $\kappa$ -Minkowski, in order to establish the form of the action of these translation generators on products of functions of the  $\kappa$ -Minkowski coordinates, the structure which is codified in the coproduct  $\Delta P_j^R$ , one can simply observe that

$$\begin{aligned} P_j^R \Omega_R(e^{ikx}) \Omega_R(e^{ipx}) &= -i \Omega_R(\partial_j e^{i(k\dot{+}p)x}) = \\ &= -i \Omega_R((k\dot{+}p)_j e^{i(k\dot{+}p)x}) = \\ &= [P_j^R \Omega_R(e^{ikx})][\Omega_R(e^{ipx})] + [e^{-\lambda P_0^R} \Omega_R(e^{ikx})][P_j^R \Omega_R(e^{ipx})], \end{aligned} \quad (1.50)$$

where  $k\dot{+}p \equiv (k_0 + p_0, \vec{k} + e^{-\lambda k_0} \vec{p})$ . This is conventionally described by the symbolic notation

$$\Delta P_j^R = P_j^R \otimes 1 + e^{-\lambda P_0^R} \otimes P_j^R. \quad (1.51)$$

Following an analogous procedure one can derive

$$\Delta P_0^R = P_0^R \otimes 1 + 1 \otimes P_0^R, \quad (1.52)$$

i.e., while for space translations one has a nontrivial coproduct, for time translations the coproduct is trivial.

Using the full machinery of the mathematics of Hopf algebras one can verify that the quadruplet of operators  $P_\mu^R$  does give rise to a genuine Hopf algebra of translation-like symmetry transformations.

### 1.6.2 Rotations

Following the same idea that allows us to introduce translations in  $\kappa$ -Minkowski, we attempt now to obtain a 7-generators Hopf algebra, describing four translation-like operators and three rotation-like generators.

For what concerns the translations we have found that an acceptable Hopf-algebra description was obtained by straightforward “quantization” of the classical translations: the  $P_\mu^R$  translations were just obtained from the commutative-spacetime translations through the  $\Omega_R$  Weyl map. Also for rotations this strategy turns out to be successful:

$$M_j^R \Omega_R(f) = \Omega_R(M_j f) = \Omega_R(-i\epsilon_{jkl} x_k \partial_l f). \quad (1.53)$$

And, while for the (spatial) translations one finds nontrivial coproduct, the coproduct of rotations is trivial:

$$\Delta M_j = M_j \otimes 1 + 1 \otimes M_j; \quad (1.54)$$

it is also straightforward to verify that

$$[M_j, M_k] = i\epsilon_{jkl} M_l. \quad (1.55)$$

Therefore the triplet  $M_j$  forms a 3-generator Hopf algebra that is completely undeformed (classical) both in the algebra and coalgebra sectors. (Using the intuitive description introduced earlier this is a trivial rotation Hopf algebra, whose structure could be equally well captured by the standard Lie algebra of rotations.)

There is therefore a difference between the translations sector and the rotations sector. Both translations and rotations can be realized as straightforward (up to ordering) quantization of their classical actions, but while for rotations even the coalgebraic properties are classical (trivial coalgebra) for the translations we found a nontrivial coalgebra sector.

Our translations and rotations can be put together straightforwardly to obtain a 7-generator translations-rotations symmetry Hopf algebra. It is sufficient to observe that

$$[M_j, P_\mu^R] \Omega(e^{ikx}) = \epsilon_{jkl} \Omega([-x_k \partial_\mu + \partial_\mu x_k] \partial_l e^{ikx}) = \delta_{\mu k} \epsilon_{jkl} \Omega(\partial_l e^{ikx}) \quad (1.56)$$

from which it follows that

$$[M_i, P_j^R] = i\epsilon_{ijk} P_k^R, \quad [M_i, P_0^R] = 0, \quad (1.57)$$

i.e. the action of rotations on energy-momentum is undeformed. Accordingly, the generators  $M_j$  can be represented as differential operators over energy-momentum space in the familiar way:  $M_j = -i\epsilon_{jkl}P_k\partial_{P_l}$ .

### 1.6.3 Boosts

In the analysis of translations and rotations in  $\kappa$ -Minkowski we have already encountered two different situations: rotations are essentially classical in all respects, while translations have a “classical” action (straightforward  $\Omega$ -map “quantization” of the corresponding classical action) but have nontrivial coalgebraic properties (nontrivial coproduct). Of course, the fact that some symmetry transformations in a noncommutative spacetime allow “classical” description (through the Weyl map) is not to be expected in general. In general one can only require that the results should reproduce the familiar ones for commutative Minkowski in the limit of vanishing noncommutativity parameters ( $\lambda \rightarrow 0$ ). As we now intend to include also boosts, and obtain 10-generator symmetry algebras, we encounter another possibility: for boosts non only the coalgebra sector is nontrivial but even the action cannot be obtained by “quantization” of the classical action.

The “classical” boosts  $N_j^R$  should have action

$$N_j^R\Omega_R(f) = \Omega_R(N_j f) = \Omega_R(i[x_0\partial_j - x_j\partial_0]f). \quad (1.58)$$

And actually it is easy to see (and it is obvious) that these boosts combine with the rotations  $M_j^R$  to close the (undeformed) Lorentz algebra, and that adding also the translations  $P_\mu^R$  one obtains the undeformed Poincaré algebra. However, these algebras cannot be extended (by introducing a suitable coalgebra sector) to obtain a Hopf algebra of symmetries of theories in our noncommutative  $\kappa$ -Minkowski spacetime. In particular, one finds an inconsistency in the coproduct of these boosts  $N_j^R$ , which signals an obstruction originating from an inadequacy in the description of the action of boosts on (noncommutative) products of  $\kappa$ -Minkowski functions. The problem is that  $\Delta(N_j^R)$  would not be an element of the algebraic tensor product, i.e. it is not a function only of the elements  $M$ ,  $N$ ,  $P$ .

Since the “classical” choice  $N_j^R$  is inadequate there are two possible outcomes: either there is no 10-generator symmetry-algebra extension of the 7-generator symmetry algebra  $(P_\mu^R, M_j)$  or the 10-generator symmetry-algebra extension exists but requires nonclassical boosts. The latter is true.

The generators of the needed modified boost action,  $\mathcal{N}_j$ , are found through a rather tedious analysis which can be found in literature [20] and we do not report in detail here. One starts by observing that, by imposing that the deformed boost generator  $\mathcal{N}_j$  (although possibly having a nonclassical action)

transform as a vector under spatial rotations, the most general form of  $\mathcal{N}_j$  is

$$\begin{aligned}\mathcal{N}_j\Omega(\Phi(x)) &= \Omega\{[ix_0A(-i\partial_x)\partial_j + \lambda^{-1}x_jB(-i\partial_x) + \\ &- \lambda x_lC(-i\partial_x)\partial_l\partial_j - i\epsilon_{jkl}x_kD(-i\partial_x)\partial_l]\Phi(x)\},\end{aligned}$$

where  $A, B, C, D$  are unknown functions of  $P_\mu^R$  (in the classical limit  $A = i$ ,  $D = 0$ ; moreover, as  $\lambda \rightarrow 0$  one obtains the classical limit if  $\lambda C \rightarrow 0$  and  $B \rightarrow \lambda P_0$ ).

Imposing that in the formula above

$$\mathcal{N}_j^R[\Omega(e^{ikx})\Omega(e^{ipx})] = [\mathcal{N}_{(1),j}^R\Omega(e^{ikx})][\mathcal{N}_{(2),j}^R\Omega(e^{ipx})]$$

it should be possible to write  $\mathcal{N}_{(1),j}$  and  $\mathcal{N}_{(2),j}$  in terms of generators of the Hopf algebra, one clearly obtains some constraints on the function  $A, B, C, D$ . The final result is

$$\mathcal{N}_j^R\Omega_R(f) = \Omega_R([ix_0\partial_j + x_j(\frac{1 - e^{2i\lambda\partial_0}}{2\lambda} - \frac{\lambda}{2}\nabla^2) - \lambda x_l\partial_l\partial_j]f). \quad (1.59)$$

It is easy to verify that the Hopf algebra  $(P_\mu^R, M_j^R, \mathcal{N}_j^R)$  satisfy all the requirements for a candidate symmetry algebra for theories in  $\kappa$ -Minkowski spacetime.

## 1.7 Mass Casimir and deformed Klein-Gordon equation

Of course, the fact that one replaces the “classical” Poincaré Lie algebra with the “quantum” deformed-Poincaré Hopf algebra has some striking consequences. For what concerns the search of a description of scalar fields the key ingredient is to find the “mass Casimir” in the quantum version, i.e. a differential operator  $\square_\lambda$ , which in the classical limit reduces to the D’Alembert operator  $\square = \partial^2$ , such that the action

$$S(\Phi) = \int d^4x \Phi(\square_\lambda - M^2)\Phi \quad (1.60)$$

is invariant under the realization of Hopf-algebra symmetry we have constructed  $(P_\mu^R, M_j^R, \mathcal{N}_j^R)$ . We therefore must verify that, for some choice of  $\square_\lambda, [\square_\lambda, T] = 0$  for every  $T$  in the Hopf algebra.



Guided by the intuition that  $\square_\lambda$  should be a scalar with respect to  $(P_\mu^R, M_j^R, \mathcal{N}_j^R)$  transformations, one is led to the proposal

$$\square_\lambda = \left( \frac{2}{\lambda} \sinh\left(\frac{\lambda P_0^R}{2}\right) \right)^2 - e^{\lambda P_0^R} (\vec{P}^R)^2. \quad (1.61)$$

In fact, it is easy to verify that with this choice of  $\square_\lambda$  the action (1.60) is invariant under the  $(P_\mu^R, M_j^R, \mathcal{N}_j^R)$  transformations. Therefore, we have finally managed to construct an action describing free scalar fields in  $\kappa$ -Minkowski that enjoys 10-generator (Hopf-algebra) symmetries  $(P_\mu^R, M_j^R, \mathcal{N}_j^R)$ .

Since the Casimir  $\square_\lambda$  is a scalar, we can ask a free scalar field theory to satisfy the Klein-Gordon-like motion equation with respect to the deformed D'Alembert operator  $\square_\lambda$ :

$$(\square_\lambda - M^2)\Phi(x) = 0. \quad (1.62)$$

This equation of motion can be obtained from the variation of the action (1.60), as shown in [17].

## Chapter 2

# Noether analysis with four-dimensional differential calculus

The new result reported in this thesis is a Noether analysis of translation symmetries in  $\kappa$ -Minkowski using a five-dimensional bicovariant differential calculus. In preparation for that derivation, which is the subject of the next chapter, we find useful to review briefly the known result for the corresponding Noether analysis of [17] with the four-dimensional differential calculus. We will present the analysis for the massless case and show at the end of the chapter how, for a complex plane wave field, the expression for the translation-symmetry conserved charges obtained gives a non-linear Planck-scale modification of the dispersion relation.

### 2.1 Translation transformation and 4D differential calculus

Before [17], previous attempts to derive translation-symmetry conserved charges in  $\kappa$ -Minkowski noncommutative spacetime failed due to the adoption of a rather naive description of translation transformation, which in particular did not take into account the properties of the noncommutative  $\kappa$ -Minkowski differential calculus. In [17] it was shown that by taking properly into account the properties of the differential calculus one encounters no obstruction in following all the steps of the Noether analysis and one obtains an explicit formula relating fields and energy-momentum charges.

In order to characterize translation transformations, if one concentrates on the infinitesimal translation parameters, rather than the generators, and tries to enforce in  $\kappa$ -Minkowski the view of infinitesimal translation as a map

$x_\mu \rightarrow x_\mu + \epsilon_\mu$ , as customary in the commutative limit, then one finds that the translation parameters must have nontrivial algebraic properties

$$[\epsilon_j, x_0] = i\lambda\epsilon_j, \quad [\epsilon_j, x_k] = 0, \quad [\epsilon_0, x_\mu] = 0 \quad (2.1)$$

in order to ensure that the “point”  $x + \epsilon$  still belongs to the  $\kappa$ -Minkowski space-time:

$$[x_j + \epsilon_j, x_0 + \epsilon_0] = i\lambda(\epsilon_j + x_j), \quad [x_i + \epsilon_i, x_j + \epsilon_j] = 0. \quad (2.2)$$

These algebraic relations reflect the known properties of the  $\kappa$ -Minkowski differential calculus [37] (the  $\epsilon$ ’s describe the difference between the coordinates of two spacetimes points and are therefore related to the  $dx$ ’s of the differential calculus).

In order to perform the Noether analysis it is necessary to describe the action of translation transformations on the fields  $f$ , which will be of the type  $f \rightarrow f + df$ . The definition of  $df$  has not to be treated as a freedom allowed by the formalism: the exterior derivative operator  $d$  must of course satisfy the Leibnitz rule

$$d(f \cdot g) = df \cdot g + f \cdot dg. \quad (2.3)$$

If we consider the translation transformation in the commutative case, we have  $df = i[P^\mu f(x)]\epsilon_\mu$ . If one tries to extend this definition to  $\kappa$ -Minkowski just substituting the commutative translation generators with the  $\kappa$ -Poincaré ones, the  $P_\mu$  translation generators of the Majid-Ruegg  $\kappa$ -Poincaré basis (1.29), (1.30), Leibnitz rule (2.3) cannot be satisfied due to the nontrivial coproduct of  $P_\mu$ . It is crucial for the analysis to observe that the form of the generators  $P_\mu$  and the properties of the infinitesimal translation parameters  $\epsilon_\mu$  must be combined in the description of the  $df$ . And the fact that in the  $\kappa$ -Minkowski case the transformation parameters have nontrivial algebraic properties poses an ordering issue, there is in fact an infinity of different formulations of the  $df$  which all reduce to  $df = i[P^\mu f(x)]\epsilon_\mu$  in the classical-spacetime (commutative) limit.

Taking into account the  $\epsilon$ ’s algebraic properties (2.1) and the coalgebra of  $\kappa$ -Poincaré translation generators, one easily finds that the requirement (2.3) singles out the formula

$$df = i\epsilon_\mu P^\mu f(x). \quad (2.4)$$

It is through this formula, involving both generators and infinitesimal parameters, that one can truly characterize the translation transformations. The ex-

clusive knowledge of the translation generators properties is clearly insufficient.

## 2.2 Noether analysis for the massless case

It is easy to verify that this improved description of translation transformations actually allows to complete the Noether analysis, thereby obtaining the energy-momentum charges.

We perform the Noether analysis for a theory of massless free scalar fields solutions of the following much studied [18, 19, 20], Klein-Gordon-like equation

$$C_\lambda(P_\mu)\Phi = \left[ \left( \frac{2}{\lambda} \sinh\left(\frac{\lambda P_0}{2}\right) \right)^2 - e^{\lambda P_0} (\vec{P})^2 \right] \Phi = 0. \quad (2.5)$$

This equation of motion can be derived from the following action

$$S[\Phi] = \int d^4x \mathcal{L}[\Phi(x)] = -\frac{1}{2} \int d^4x \tilde{P}^\mu \Phi \tilde{P}_\mu \Phi, \quad (2.6)$$

where we introduced the compact notation  $\tilde{P}_\mu$ ,

$$\tilde{P}_0 = \frac{2}{\lambda} \sinh \frac{\lambda}{2} P_0 \quad \tilde{P}_i = P_i e^{\frac{\lambda}{2} P_0}, \quad (2.7)$$

which also allows to rewrite  $C_\lambda(P_\mu)$  as  $\tilde{P}_\mu \tilde{P}^\mu$ . The most general solution of eq. (2.5) can be written as

$$\Phi(x) = \int d^4k \tilde{f}(k_0, \vec{k}) e^{i\vec{k} \cdot \vec{x}} e^{-k_0 x_0} \delta(C_\lambda(k_\mu)). \quad (2.8)$$

If we now consider the variation  $\delta\Phi$  applied to the field  $\Phi$

$$\Phi \rightarrow \Phi' = \Phi + \delta\Phi, \quad (2.9)$$

the action then varies according to

$$\begin{aligned} \delta S[\Phi] &= \int d^4x (\mathcal{L}[\Phi'(x')] - \mathcal{L}[\Phi(x)]) = \\ &= \frac{1}{2} \int d^4x \left\{ e^{\frac{\lambda P_0}{2}} \left[ (\tilde{P}_\mu \tilde{P}^\mu \Phi) \delta\Phi \right] + e^{-\frac{\lambda P_0}{2}} \left[ \delta\Phi (\tilde{P}_\mu \tilde{P}^\mu \Phi) \right] \right\} + \end{aligned}$$

$$+ \int d^4x \left\{ -\frac{1}{2} \tilde{P}^\mu \left[ e^{\frac{\lambda P_0}{2}} \tilde{P}_\mu \Phi \delta \Phi + \delta \Phi e^{-\frac{\lambda P_0}{2}} \tilde{P}_\mu \Phi \right] + \mathcal{L}[\Phi(x')] - \mathcal{L}[\Phi(x)] \right\}, \quad (2.10)$$

where we also used the observation that

$$\tilde{P}_\mu[f(x)g(x)] = [\tilde{P}_\mu f(x)][e^{\frac{\lambda}{2}P_0}g(x)] + [e^{-\frac{\lambda}{2}P_0}f(x)][\tilde{P}_\mu g(x)] \quad (2.11)$$

for any field  $f(x)$  and  $g(x)$ .

In (2.10) there are two separated integrals: the first integral represents the action variation that gives the equation of motion, while the second integral gives the border terms in the action variation from which we obtain the conserved currents, once imposed the equation of motion. This second integral contains itself two terms: the first one originates from the variation  $\delta\Phi \equiv \Phi'(x) - \Phi(x)$  of the fields, the second one from the variation  $d\Phi \equiv \Phi(x') - \Phi(x)$  of the field coordinates. We remind that, by definition of scalar field, holds

$$0 = \Phi'(x') - \Phi(x) = |\Phi'(x') - \Phi(x')| - |\Phi(x') - \Phi(x)| \rightarrow \delta\Phi = -d\Phi \quad (2.12)$$

whit the approximation  $\delta\Phi(x') \equiv \Phi'(x') - \Phi(x') \simeq \Phi'(x) - \Phi(x) \equiv \delta\Phi$ , in order to consider variations at the first order.

Using the equation of motion (2.5) one easily obtains the following description of the total variation of our action (2.6) under a translation transformation ( $x \rightarrow x + dx$  and  $f \rightarrow f + df$ ):

$$\begin{aligned} \delta S[\Phi] &= -\frac{1}{2} \int d^4x \left\{ \epsilon^\mu \left( (\tilde{P}_\alpha e^{-\lambda P_0 \delta_{\mu j}} \Phi) (\tilde{P}^\alpha P_\mu \Phi) + (P_\mu \tilde{P}_\alpha \Phi) \tilde{P}^\alpha \Phi \right) \right\} + \\ &- \int d^4x \{ \epsilon^\mu P_\mu \mathcal{L} \} = -\frac{1}{2} \epsilon^\mu \int d^4x \tilde{P}^\alpha [(\tilde{P}_\alpha e^{(-\delta_{\mu j} + \frac{1}{2})\lambda P_0} \Phi) (P_\mu \Phi) + \\ &+ (P_\mu \Phi) \tilde{P}_\alpha e^{-\frac{\lambda P_0}{2}} \Phi] - \epsilon^\mu \int d^4x \{ P_\mu \mathcal{L} \} = \\ &= \int d^4x \{ \epsilon^\mu \tilde{P}^\nu J_{\mu\nu} \}, \end{aligned} \quad (2.13)$$

where

$$J_{j\mu} = -\frac{1}{2} (\tilde{P}_j e^{(-\delta_{\mu j} + \frac{1}{2})\lambda P_0} \Phi) (P_\mu \Phi) + \frac{1}{2} (P_\mu \Phi) \tilde{P}_j e^{-\frac{\lambda P_0}{2}} \Phi - \delta_{\mu j} P_j \tilde{P}_j^{-1} \mathcal{L}, \quad (2.14)$$

$$J_{0\mu} = -\frac{1}{2}(\tilde{P}_0 e^{(-\delta_{\mu j} + \frac{1}{2})\lambda P_0} \Phi)(P_\mu \Phi) + \frac{1}{2}(P_\mu \Phi) \tilde{P}_0 e^{-\frac{\lambda P_0}{2}} \Phi - \delta_{\mu 0} P_0 \tilde{P}_0^{-1} \mathcal{L}. \quad (2.15)$$

Performing a 3D spatial integration of the component  $J_{0\mu}$  and evaluating the charges on the solution of the equation of motion, whose general form is given in (2.8), one easily finds the following expression for the charges carried by the solutions of the equation of motion:

$$Q_\mu = \int d^3x J_{0\mu} = \frac{1}{2} \int d^4p e^{3\lambda P_0} p_\mu \tilde{\Phi}(p_0, \vec{p}) \tilde{\Phi}(-p_0, -e^{\lambda P_0} \vec{p}) \frac{p_0}{|p_0|} \delta(C_\lambda(p_\mu)). \quad (2.16)$$

The fact that these energy-momentum charges  $Q_\mu$ , computed by 3D spatial integration of the  $J_{0\mu}$ , are indeed time independent confirms that the Noether analysis has been successful.

It is rather clear from the form of (2.16) that the energy-momentum relation is Planck-scale-( $\lambda$ -)deformed with respect to the special-relativistic (Poincaré-Lie-algebra) limit. Let us consider for example a “regularized plane wave solution” whose Fourier transform is

$$\tilde{\Phi}(k) = \frac{2\sqrt{|\vec{k}|}\theta(k_0)\delta(\vec{k} - \vec{p})}{\sqrt{V}}. \quad (2.17)$$

It is easy to see that the field  $\Phi(x)$  can be written as

$$\begin{aligned} \Phi(x) &= \int d^4k \frac{2\sqrt{|\vec{k}|}\theta(k_0)\delta(\vec{k} - \vec{p})}{\sqrt{V}} e^{i\vec{k}\cdot\vec{x}} e^{-ik_0x_0} \delta(C_\lambda(k_0, \vec{k}) - m^2) = \\ &= \int \frac{d^4k}{2|\vec{k}|} \frac{2\sqrt{|\vec{k}|}\delta(\vec{k} - \vec{p})}{\sqrt{V}} e^{i\vec{k}\cdot\vec{x}} e^{-ik_0x_0} \delta(k_0 - k_0^+) = \\ &= \frac{1}{\sqrt{V}|\vec{p}|} e^{i\vec{p}\cdot\vec{x}} e^{-ip_0^+x_0}. \end{aligned} \quad (2.18)$$

For a complex scalar classic field  $\Phi$ , solution of  $C_\lambda(k)\Phi = 0$  on  $\kappa$ -Minkowski

$$\Phi(x) = \int d^4k \tilde{\Phi}(k) \delta(C_\lambda(k)) e^{i\vec{k}\cdot\vec{x}} e^{-ik_0x_0}$$

holds the condition

$$\left(\tilde{\Phi}(k_0, \vec{k})\right)^* = \left(\tilde{\Phi}^*(-k_0, -\vec{k}e^{\lambda k_0})\right)^* e^{3\lambda k_0} \quad (2.19)$$

that allows us to rewrite the charges as

$$Q_\mu = \frac{1}{2} \int d^4k |\tilde{\Phi}(k_0, \vec{k})|^2 k_\mu \frac{k_0}{|k_0|} \delta(C_\lambda(k)). \quad (2.20)$$

Using these results and the solutions of  $\delta(C_\lambda(k))$

$$\begin{aligned} \delta(C_\lambda(k)) &= \delta\left(\left(\frac{2}{\lambda} \sinh \frac{\lambda k_0}{2}\right)^2 - |\vec{k}|^2 e^{\lambda k_0}\right) = \\ &= \frac{1}{2|\vec{k}|} (\delta(k_0 - k_0^+) + \delta(k_0 - k_0^-)), \end{aligned} \quad (2.21)$$

where

$$\begin{aligned} k_0^+ &= \frac{1}{\lambda} \ln \left( \frac{1}{1 - (\lambda |\vec{k}|)} \right) \\ k_0^- &= \frac{1}{\lambda} \ln \left( \frac{1}{1 + (\lambda |\vec{k}|)} \right), \end{aligned} \quad (2.22)$$

we are now ready to compute the charges

$$\begin{aligned} (Q_0, \vec{Q}) &= \frac{1}{2} \int d^4k \left| \frac{2\sqrt{|\vec{k}|} \theta(k_0) \delta(\vec{k} - \vec{p})}{\sqrt{V}} \right|^2 \frac{k_0}{|k_0|} (k_0, \vec{k}) \delta(C_\lambda(k)) = \\ &= \frac{1}{2} \int d^4k \frac{|2\sqrt{|\vec{k}|}|^2}{2|\vec{k}|} \frac{k_0}{|k_0|} (k_0, \vec{k}) \delta(k_0 - k_0^+) \delta(\vec{k} - \vec{p}) = \\ &= \int d^3k \frac{k_0}{|k_0|} (k_0^+, \vec{k}) \delta(\vec{k} - \vec{p}) = (p_0^+, \vec{p}) \end{aligned} \quad (2.23)$$

which are on shell with respect to the Casimir, i.e.  $C_\lambda(Q_\mu) = 0$ . Therefore we can write the dispersion relation of the charges  $Q_\mu$  associated to the field (2.2):

$$\left(\frac{2}{\lambda}\right)^2 \sinh^2\left(\frac{\lambda Q_0}{2}\right) - e^{\lambda Q_0} Q_i^2 = 0. \quad (2.24)$$

Of course, in the special-relativistic limit,  $\lambda \rightarrow 0$ , one recovers the standard energy-momentum relation  $Q_o^2 - Q_i^2 = 0$  (for our massless fields), but in general some  $\lambda$ -dependent corrections are present.

## Chapter 3

# Noether analysis with five-dimensional bicovariant differential calculus

In the previous chapter we saw how taking into account the properties of the noncommutative  $\kappa$ -Minkowski differential calculus turns out to be a key point to obtain conserved translation-symmetry charges. While in the commutative case there is only one (natural) differential calculus involving the conventional derivatives, in the  $\kappa$ -Minkowski case (and in general in a noncommutative spacetime) the introduction of a differential calculus is a more complex problem and, in particular, it is not unique. In our analysis of this chapter we focus on a possible choice of differential calculus in  $\kappa$ -Minkowski: the five-dimensional (5D) bicovariant differential calculus introduced by Sitarz in [22]. The vector fields corresponding to this differential calculus has in fact special covariance properties: they transform under  $\kappa$ -Poincaré in the same way that the ordinary derivatives (i.e. the vector fields associated to the differential calculus in the commutative Minkowski space) transform under Poincaré. The fact that this calculus is bi-covariant under the action of the full  $\kappa$ -Poincaré algebra<sup>1</sup> motivated some authors (see, e.g., [27, 28]) to argue that the charges associated to the translation symmetry and the translation-symmetry relation derived from this calculus should have the same properties of the corresponding charges in the classical Minkowski spacetime.

Before this thesis work, the possibility to use the properties of the five-dimensional differential calculus and work exclusively on the noncommutative  $\kappa$ -Minkowski spacetime in performing a Noether analysis had never been considered. An attempt to obtain charges from the 5D differential calculus was present in the literature ([27]) but relied on an uncontrolled map between the

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<sup>1</sup>We remind that the differential calculus used in the previous chapter and proposed by [21] is a four-dimensional translational invariant calculus, but is not covariant, in the sense of Sitarz [22], under the action of the full  $\kappa$ -Poincaré algebra.



noncommutative spacetime theory here of interest and a commutative spacetime theory (the results reported in this chapter expose the inadequacy of that proposed correspondence). Working exclusively on the noncommutative spacetime, one obtains explicit formulas relating fields and energy-momentum charges, that turn out to be non-classical functionals of the fields with a non-trivial  $\lambda$  dependence.

In the first section we introduce the five-dimensional differential calculus, whose construction following the Sitarz procedure is reported in Appendix B, and the proper translation generators with their co-algebra sector and the suitable commutation relations between fields and one-forms derived from the 5D differential calculus. In the second part of the chapter, equipped with all these tools, we perform all the steps of the Noether analysis, in analogy with the previous chapter.

### 3.1 Bicovariant differential calculus on $\kappa$ -Minkowski

In the commutative case there is only one “natural” differential calculus, which involves the ordinary derivatives. In this case, the exterior derivative operator  $d$  of a commutative function  $f(x)$  is the usual one:

$$df(x) = dx^\mu \partial_\mu f(x) = i dx^\mu P_\mu f(x) \quad (3.1)$$

where we have expressed the vector fields  $\partial_\mu$  in terms of the standard translation generators  $P_\mu = -i\partial_\mu$ . In this way it is clear that the  $\partial_\mu$  transform *covariantly* under the standard Lorentz algebra (generated by  $M_j, N_j$ ):

$$\begin{aligned} [M_j, P_0] &= 0, & [M_j, P_k] &= i\epsilon_{jkl} P_l, \\ [N_j, P_0] &= iP_j, & [N_j, P_k] &= i\delta_{jk} P_0. \end{aligned} \quad (3.2)$$

In the case of  $\kappa$ -Minkowski, instead, the choice of a differential calculus is not unique. We introduce below a possible choice of differential calculus in  $\kappa$ -Minkowski, the 5D differential calculus. In this differential calculus the exterior derivative operator  $d$  of a generic  $\kappa$ -Minkowski element  $F(\hat{x}) = \Omega(f(x))$  can be written in the form<sup>2</sup>:

$$dF(\hat{x}) = d\hat{x}^A \hat{P}_A(P) F(\hat{x}), \quad A = 0, \dots, 4, \quad (3.3)$$

---

<sup>2</sup>We use greek letters for indexes running over  $(\alpha, \mu = 0, 1, 2, 3)$ , small latin letters for indexes running over  $(j, k = 1, 2, 3)$  and capital latin letters for indexes running over  $(A, B = 0, 1, 2, 3, 4)$ .

where the operators  $\hat{P}_0, \hat{P}_1, \hat{P}_2, \hat{P}_3$  form a basis for the translation generators of  $\kappa$ -Poincaré, while  $\hat{P}_4$  is connected with the Casimir  $C_\lambda(P)$ . The operators  $\hat{P}_A$  are defined

$$\hat{P}_0 = \frac{1}{\lambda}(\sinh \lambda P_0 + \frac{\lambda^2}{2}P^2 e^{\lambda P_0}) \quad (3.4)$$

$$\hat{P}_i = P_i e^{\lambda P_0} \quad i = 1, 2, 3 \quad (3.5)$$

$$\hat{P}_4 = \frac{1}{\lambda}(\cosh \lambda P_0 - 1 - \frac{\lambda^2}{2}P^2 e^{\lambda P_0}) = \frac{\lambda}{2}m^2, \quad (3.6)$$

where  $P_\mu$  denotes again (as in chapter 1 and 2) the translation generators of the Majid-Ruegg  $\kappa$ -Poincaré basis, whose action on a right-ordered function of  $\kappa$ -Minkowski is  $P_\mu(e^{ik\hat{x}}e^{-ik_0\hat{x}_0}) = k_\mu(e^{ik\hat{x}}e^{-ik_0\hat{x}_0})$ . The commutation relations between the one-form generators  $d\hat{x}^A$  and the  $\kappa$ -Minkowski generators  $\hat{x}^\mu$  are:

$$[\hat{x}_0, d\hat{x}_4] = i\lambda d\hat{x}_0, \quad [\hat{x}_0, d\hat{x}_i] = i\lambda d\hat{x}_i, \quad [\hat{x}_0, d\hat{x}_i] = 0,$$

$$[\hat{x}_i, d\hat{x}_4] = [\hat{x}_i, d\hat{x}_0] = -i\lambda d\hat{x}_i, \quad [\hat{x}_i, d\hat{x}_j] = i\lambda \delta_{ij}(d\hat{x}_4 - d\hat{x}_0). \quad (3.7)$$

The introduction of such a 5D calculus in our 4D spacetime may at first appear to be surprising, but it can be naturally introduced on the basis of the fact that the  $\kappa$ -Poincaré/ $\kappa$ -Minkowski framework can be obtained (and was indeed originally obtained [35]) by Inönü-Wigner contraction of a 5D q-deformed anti-De Sitter algebra. The fifth one-form generator is here denoted by “ $d\hat{x}^4$ ”, but this is of course only a formal notation, since there is no fifth  $\kappa$ -Minkowski coordinate  $\hat{x}^4$ . And the peculiar role of  $d\hat{x}^4$  in this differential calculus is also codified in the fact that the last component  $\hat{P}_4(P)$  is essentially the Casimir (1.34) of  $\kappa$ -Poincaré:

$$\hat{P}_4(P) = \frac{\lambda}{2}C_\lambda(P). \quad (3.8)$$

This differential calculus is characterized by interesting transformation properties under the action of the Lorentz sector of  $\kappa$ -Poincaré. In fact taking into account (1.30) one finds that:

$$[M_j, \hat{P}_0] = 0, \quad [M_j, \hat{P}_k] = i\epsilon_{jkl}\hat{P}_l, \quad [M_j, \hat{P}_4] = 0,$$

$$[N_j, \hat{P}_0] = i\hat{P}_j, \quad [N_j, \hat{P}_k] = i\delta_{jk}\hat{P}_0, \quad [N_j, \hat{P}_4] = 0. \quad (3.9)$$

Thus the operators  $\hat{P}_\mu$  transform under  $\kappa$ -Poincaré action in the same way as the  $P_\mu$  operators transform under the standard Poincaré action, while  $\hat{P}_4(P)$  is invariant.

This differential calculus originates in [22] by the request that it remains invariant under the action of the  $\kappa$ -Poincaré action, i.e. the commutation relations (3.7) that characterize it remain invariant<sup>3</sup> under the action of the  $\kappa$ -Poincaré generators; practically, one seeks some  $d\hat{x}^A$  such that their commutator with the  $\kappa$ -Minkowski coordinates  $\hat{x}^\mu$

$$[d\hat{x}^A, \hat{x}^\mu] = v_\rho^{A\mu} d\hat{x}^\rho, \quad (3.10)$$

for some numbers  $v_\rho^{A\mu}$ , are invariant in the sense

$$T[d\hat{x}^A, \hat{x}^\mu] = v_\rho^{A\mu} T d\hat{x}^\rho, \quad (3.11)$$

where  $T$  denotes any one of the  $\kappa$ -Poincaré generators  $(P_\mu, M_j, N_j)$ . A differential calculus in which the commutation relations between the one-form generators and the  $\kappa$ -Minkowski generators remain *invariant* under the action of the symmetry algebra ( $\kappa$ -Poincaré in our case), is called “bicovariant” differential calculus. In [38] it was claimed that the 5D differential calculus of Sitarz is the unique bicovariant one with respect to the left action of  $\kappa$ -Poincaré group.

In order to perform our Noether analysis of the translation sector of  $\kappa$ -Poincaré with the 5D differential calculus it is convenient to first derive formulas for the coproducts of the operators  $\hat{P}_A$  and their commutation relations with the time-to-the-right-ordered plane wave basis of  $\kappa$ -Minkowski.

The form of the coproducts of  $\hat{P}_A$  is easily obtained exploiting the relationship (3.4), (3.5) and (3.6) between  $\hat{P}_A$  and  $P_\mu$  and the fact that we have already provided formulas, (1.32), for the coproduct of  $P_\mu$ . Taking into account the homomorphism property of the coproduct map (1.15), one finds that

$$\Delta(\hat{P}_0) = \hat{P}_0 \otimes e^{\lambda P_0} + e^{-\lambda P_0} \otimes \hat{P}_0 + \lambda P_i \otimes \hat{P}_i \quad (3.12)$$

$$\Delta(\hat{P}_i) = \hat{P}_i \otimes e^{\lambda P_0} + 1 \otimes \hat{P}_i \quad (3.13)$$

---

<sup>3</sup>The demonstration is reported in detail in Appendix A

$$\Delta(\hat{P}_4) = \hat{P}_4 \otimes e^{\lambda P_0} - e^{-\lambda P_0} \otimes \hat{P}_0 - \lambda P_i \otimes \hat{P}_i + 1 \otimes \left( \frac{e^{\lambda P_0} - 1}{\lambda} \right). \quad (3.14)$$

For the commutation relations between the time-to-the-right-ordered plane waves  $e^{ik \cdot \hat{x}} e^{-k_0 \hat{x}_0}$  and the  $d\hat{x}_A$  elements of the 5D differential calculus one finds

$$e^{ik \cdot \hat{x}} e^{-k_0 \hat{x}_0} d\hat{x}_0 = [(\lambda \hat{P}_0 + e^{-\lambda P_0}) d\hat{x}_0 + \lambda \hat{P}_i d\hat{x}_i + (\lambda \hat{P}_4 + 1 - e^{-\lambda P_0}) d\hat{x}_4] e^{ik \cdot \hat{x}} e^{-k_0 \hat{x}_0} \quad (3.15)$$

$$e^{ik \cdot \hat{x}} e^{-k_0 \hat{x}_0} d\hat{x}_i = [\lambda P_i d\hat{x}_0 + d\hat{x}_i - \lambda P_i d\hat{x}_4] e^{ik \cdot \hat{x}} e^{-k_0 \hat{x}_0} \quad (3.16)$$

$$e^{ik \cdot \hat{x}} e^{-k_0 \hat{x}_0} d\hat{x}_4 = [\lambda \hat{P}_0 d\hat{x}_0 + \lambda \hat{P}_i d\hat{x}_i + (\lambda \hat{P}_4 + 1) d\hat{x}_4] e^{ik \cdot \hat{x}} e^{-k_0 \hat{x}_0}. \quad (3.17)$$

## 3.2 Noether analysis

### 3.2.1 Leibnitz rule

With the tools introduced in the previous section we are now ready to perform the Noether analysis of  $\kappa$ -Poincaré translation symmetry for  $\kappa$ -Minkowski.

The exterior derivative operator  $d$  of a general element of  $\kappa$ -Minkowski, defined in (3.3), must of course satisfy the Leibnitz rule with respect of the coproducts of the translation generators  $\hat{P}_A$ . Let us show this using the commutation relations (3.15)-(3.17) between the one-forms generators and the time-to-the-right-ordered plane waves and remembering that

$$\hat{P}_0 + \hat{P}_4 = \frac{e^{\lambda P_0} - 1}{\lambda}.$$

We have:

$$\begin{aligned}
(d\Psi)\Phi + \Psi(d\Phi) &= (dx^0\hat{P}_0\Psi)\Phi + \Psi(dx^0\hat{P}_0\Phi) + (dx^i\hat{P}_i\Psi)\Phi + \\
&+ \Psi(dx^i\hat{P}_i\Phi) + (dx^4\hat{P}_4\Psi)\Phi + \Psi(dx^4\hat{P}_4\Phi) = \\
&= (dx^0\hat{P}_0\Psi)\Phi + [(\lambda\hat{P}_0 + e^{-\lambda P_0})dx^0\Psi\hat{P}_0\Phi + \lambda\hat{P}_i dx^i\Psi\hat{P}_0\Phi + \\
&+ (\lambda\hat{P}_4 + 1 - e^{-\lambda P_0})dx^4\Psi\hat{P}_0\Phi] + (dx^i\hat{P}_i\Psi)\Phi + \\
&+ [\lambda\hat{P}_i dx^0\Psi\hat{P}_i\Phi + dx^i\Psi\hat{P}_i\Phi - \lambda\hat{P}_i dx^4\Psi\hat{P}_i\Phi] + (dx^4\hat{P}_4\Psi)\Phi + \\
&+ [\lambda\hat{P}_0 dx^0\Psi\hat{P}_4\Phi + \lambda\hat{P}_i dx^i\Psi\hat{P}_4\Phi + (\lambda\hat{P}_4 + 1)dx^4\Psi\hat{P}_4\Phi] = \\
&= dx^0[\hat{P}_0\Psi\Phi + \lambda\hat{P}_0\Psi\hat{P}_0\Phi + e^{-\lambda P_0}\Psi\hat{P}_0\Phi + \lambda\hat{P}_i\Psi\hat{P}_i\Phi + \lambda\hat{P}_0\Psi\hat{P}_4\Phi] + \\
&+ dx^i[\hat{P}_i\Psi\Phi + \Psi\hat{P}_i\Phi + \lambda\hat{P}_i\Psi\hat{P}_0\Phi + \lambda\hat{P}_i\Psi\hat{P}_4\Phi] + \\
&+ dx^4[(\lambda\hat{P}_4 + 1)\Psi(\hat{P}_0 + \hat{P}_4)\Phi - e^{-\lambda P_0}\Psi\hat{P}_0\Phi - \lambda\hat{P}_i\Psi\hat{P}_i\Phi + \hat{P}_4\Psi\Phi] = \\
&= dx^0[\hat{P}_0\Psi\Phi + \lambda\hat{P}_0\Psi(\frac{e^{\lambda P_0} - 1}{\lambda})\Phi + e^{-\lambda P_0}\Psi\hat{P}_0\Phi + \lambda\hat{P}_i\Psi\hat{P}_i\Phi] + \\
&+ dx^i[\hat{P}_i\Psi\Phi + \Psi\hat{P}_i\Phi + \lambda\hat{P}_i\Psi(\frac{e^{\lambda P_0} - 1}{\lambda})\Phi] + \\
&+ dx^4[(\lambda\hat{P}_4 + 1)\Psi(\frac{e^{\lambda P_0} - 1}{\lambda})\Phi - e^{-\lambda P_0}\Psi\hat{P}_0\Phi - \lambda\hat{P}_i\Psi\hat{P}_i\Phi + \hat{P}_4\Psi\Phi] = \\
&= dx^0[\hat{P}_0\Psi e^{\lambda P_0}\Phi + e^{-\lambda P_0}\Psi\hat{P}_0\Phi + \lambda\hat{P}_i\Psi\hat{P}_i\Phi] + \\
&+ dx^i[\hat{P}_i\Psi e^{\lambda P_0}\Phi + \Psi\hat{P}_i\Phi] + \\
&+ dx^4[\Psi(\frac{e^{\lambda P_0} - 1}{\lambda})\Phi - e^{-\lambda P_0}\Psi\hat{P}_0\Phi - \lambda\hat{P}_i\Psi\hat{P}_i\Phi + \hat{P}_4\Psi e^{\lambda P_0}\Phi] = \\
&= d(\Psi\Phi), \tag{3.18}
\end{aligned}$$

where the last equality holds with respect of the coproducts (3.12), (3.13) and (3.14). Therefore Leibnitz is satisfied.

### 3.2.2 Currents

In the following Noether analysis we assume that a massive scalar field  $\Phi(x)$  is governed by one of the most studied equation of motions in the  $\kappa$ -Minkowski literature [18, 20], i.e. the Klein-Gordon-like equation

$$C_\lambda(P_\mu)\Phi \equiv \left[ \left( \frac{2}{\lambda} \sinh \frac{\lambda}{2} P_0 \right)^2 - e^{\lambda P_0} \vec{P}^2 \right] \Phi = m^2 \Phi, \tag{3.19}$$

which can be derived from the following action

$$S[\Phi] = \int d^4x \mathcal{L}[\Phi(x)]$$

$$\mathcal{L}[\Phi(x)] = \frac{1}{2} \left( \Phi(x) C_\lambda \Phi(x) - m^2 \Phi(x) \Phi(x) \right) . \quad (3.20)$$

We remind that the operator  $C_\lambda(P_\mu)$  is the mass Casimir of the  $\kappa$ -Poincaré Hopf algebra and we find sometimes useful to also write it as  $C_\lambda = \tilde{P}_\mu \tilde{P}^\mu$  in terms of the operators

$$\tilde{P}_0 = \frac{2}{\lambda} \sinh \frac{\lambda}{2} P_0 \quad \tilde{P}_i = P_i e^{\frac{\lambda}{2} P_0} , \quad (3.21)$$

whose coproducts are given by

$$\tilde{P}_\alpha[f(x)g(x)] = [\tilde{P}_\alpha f(x)][e^{\frac{\lambda}{2} P_0} g(x)] + [e^{-\frac{\lambda}{2} P_0} f(x)][\tilde{P}_\alpha g(x)] . \quad (3.22)$$

We can now derive the total variation of our action (3.20) under a translation transformation ( $x \rightarrow x + dx$  and  $f \rightarrow f + df$ ), using eq. (3.3), (3.15)-(3.17) and the observation that, by definition of a scalar field,

$$0 = \Phi'(\hat{x}') - \Phi(\hat{x}) = [\Phi'(\hat{x}') - \Phi(\hat{x}')] - [\Phi(\hat{x}') - \Phi(\hat{x})] , \quad (3.23)$$

i.e.  $\delta\Phi = -d\Phi = -i \left( \hat{\epsilon}^0 \hat{P}_0 + \hat{\epsilon}^j \hat{P}_j + \hat{\epsilon}^4 \hat{P}_4 \right) \Phi$  (where we have identified the infinitesimal transformation parameters with the one-forms generators of the 5D differential calculus):

$$\begin{aligned}
\delta S &= \frac{1}{2} \int d^4x \left( \delta\Phi C_\lambda \Phi + \Phi C_\lambda \delta\Phi - m^2 \delta\Phi \Phi - m^2 \Phi \delta\Phi \right) = \\
&= \frac{1}{2} \int d^4x \left[ e^{\frac{\lambda P_0}{2}} \tilde{P}^0 \left( \left( \frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \Phi \hat{\epsilon}^A \hat{P}_A \Phi - \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{\epsilon}^A \hat{P}_A \Phi \right) + \right. \\
&+ \left. \hat{P}^i \left( \Phi e^{-\lambda P_0} \hat{P}_i \hat{\epsilon}^A \hat{P}_A \Phi - \hat{P}_i \Phi \hat{\epsilon}^A \hat{P}_A \Phi \right) \right] = \\
&= \frac{1}{2} \int d^4x \left\{ e^{\frac{\lambda P_0}{2}} \tilde{P}^0 \left\{ \left( \frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \left[ [(\lambda \hat{P}_0 + e^{-\lambda P_0}) \hat{\epsilon}^0 + \lambda \hat{P}_i \hat{\epsilon}^i + \right. \right. \right. \\
&+ \left. \left. (\lambda \hat{P}_4 + 1 - e^{-\lambda P_0}) \hat{\epsilon}^4] \Phi \hat{P}_0 \Phi + [\lambda e^{-\lambda P_0} \hat{P}_i \hat{\epsilon}^0 + \hat{\epsilon}^i - \lambda e^{-\lambda P_0} \hat{P}_i \hat{\epsilon}^4] \Phi \hat{P}_i \Phi + \right. \right. \\
&+ \left. \left. [\lambda \hat{P}_0 \hat{\epsilon}^0 + \lambda \hat{P}_i \hat{\epsilon}^i + (\lambda \hat{P}_4 + 1) \hat{\epsilon}^4] \Phi \hat{P}_4 \Phi \right\} + \right. \\
&- \left[ (\lambda \hat{P}_0 + e^{-\lambda P_0}) \hat{\epsilon}^0 + \lambda \hat{P}_i \hat{\epsilon}^i + (\lambda \hat{P}_4 + 1 - e^{-\lambda P_0}) \hat{\epsilon}^4 \right] \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_0 \Phi + \\
&- \left[ \lambda e^{-\lambda P_0} \hat{P}_i \hat{\epsilon}^0 + \hat{\epsilon}^i - \lambda e^{-\lambda P_0} \hat{P}_i \hat{\epsilon}^4 \right] \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_i \Phi + \\
&- \left[ \lambda \hat{P}_0 \hat{\epsilon}^0 + \lambda \hat{P}_i \hat{\epsilon}^i + (\lambda \hat{P}_4 + 1) \hat{\epsilon}^4 \right] \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_4 \Phi \left. \right\} + \\
&+ \hat{P}_i \left\{ \left[ (\lambda \hat{P}_0 + e^{-\lambda P_0}) \hat{\epsilon}^0 + \lambda \hat{P}_i \hat{\epsilon}^i + (\lambda \hat{P}_4 + 1 - e^{-\lambda P_0}) \hat{\epsilon}^4 \right] \Phi \hat{P}_i \hat{P}_0 \Phi + \right. \\
&+ \left[ \lambda e^{-\lambda P_0} \hat{P}_i \hat{\epsilon}^0 + \hat{\epsilon}^i - \lambda e^{-\lambda P_0} \hat{P}_i \hat{\epsilon}^4 \right] \Phi \hat{P}_i \hat{P}_i \Phi + \\
&+ \left[ \lambda \hat{P}_0 \hat{\epsilon}^0 + \lambda \hat{P}_i \hat{\epsilon}^i + (\lambda \hat{P}_4 + 1) \hat{\epsilon}^4 \right] \Phi \hat{P}_i \hat{P}_4 \Phi + \\
&- \left[ (\lambda \hat{P}_0 + e^{-\lambda P_0}) \hat{\epsilon}^0 + \lambda \hat{P}_i \hat{\epsilon}^i + (\lambda \hat{P}_4 + 1 - e^{-\lambda P_0}) \hat{\epsilon}^4 \right] \hat{P}_i \Phi \hat{P}_0 \Phi + \\
&- \left[ \lambda e^{-\lambda P_0} \hat{P}_i \hat{\epsilon}^0 + \hat{\epsilon}^i - \lambda e^{-\lambda P_0} \hat{P}_i \hat{\epsilon}^4 \right] \hat{P}_i \Phi \hat{P}_i \Phi + \\
&- \left. \left[ \lambda \hat{P}_0 \hat{\epsilon}^0 + \lambda \hat{P}_i \hat{\epsilon}^i + (\lambda \hat{P}_4 + 1) \hat{\epsilon}^4 \right] \hat{P}_i \Phi \hat{P}_4 \Phi \right\} \left. \right\}, \tag{3.24}
\end{aligned}$$

where we have specialized to the case of fields such that  $\tilde{P}^\mu \tilde{P}_\mu \Phi = m^2 \Phi$ , since of course we perform the Noether analysis on fields that are solutions of the equation of motion.

Thus, the variation of the Lagrangian density takes the form

$$\hat{\epsilon}^A \left( e^{\frac{\lambda P_0}{2}} \tilde{P}^0 J_{0A} + \hat{P}^i J_{iA} \right) = 0, \tag{3.25}$$

where

$$\begin{aligned}
J_{00} &= \frac{1}{2} \left\{ \left( \frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \left[ (\lambda \hat{P}_0 + e^{-\lambda P_0}) \Phi \hat{P}_0 \Phi + \lambda P_i \Phi \hat{P}_i \Phi + \lambda \hat{P}_0 \Phi \hat{P}_4 \Phi \right] + \right. \\
&\quad \left. - (\lambda \hat{P}_0 + e^{-\lambda P_0}) \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_0 \Phi - \lambda P_i \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_i \Phi - \lambda \hat{P}_0 \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_4 \Phi \right\}, \\
J_{0i} &= \frac{1}{2} \left\{ \left( \frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \left[ \lambda \hat{P}_i \Phi \hat{P}_0 \Phi + \Phi \hat{P}_i \Phi + \lambda \hat{P}_i \Phi \hat{P}_4 \Phi \right] + \right. \\
&\quad \left. - \lambda \hat{P}_i \frac{e^{-\lambda P_0}}{\lambda} \Phi \hat{P}_0 \Phi - \Phi \hat{P}_i \frac{e^{-\lambda P_0}}{\lambda} \Phi - \lambda \hat{P}_i \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_4 \Phi \right\}, \\
J_{04} &= \frac{1}{2} \left\{ \left( \frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \left[ (\lambda \hat{P}_4 + 1 - e^{-\lambda P_0}) \Phi \hat{P}_0 \Phi - \lambda P_i \Phi \hat{P}_i \Phi + \right. \right. \\
&\quad \left. \left. + (\lambda \hat{P}_4 + 1) \Phi \hat{P}_4 \Phi \right] - (\lambda \hat{P}_4 + 1 - e^{-\lambda P_0}) \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_0 \Phi + \right. \\
&\quad \left. + \lambda P_i \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_i \Phi - (\lambda \hat{P}_4 + 1) \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_4 \Phi \right\}. \tag{3.26}
\end{aligned}$$

We want now to outline that eq. (3.25), produced by the Noether analysis, guarantees the time-independence of the charges obtained from the currents (3.26). In fact, remembering that

$$e^{\frac{\lambda}{2} P_0} \tilde{P}_0 = (\hat{P}_0 + \hat{P}_4) = \frac{e^{\lambda P_0} - 1}{\lambda}, \tag{3.27}$$

eq. (3.25) constitutes a “conservation equation” in  $\kappa$ -Minkowski spacetime with the proper generator associated to temporal translations, i.e. it is the combination  $\hat{P}_0 + \hat{P}_4$  which vanishes on time-independent fields and not  $\hat{P}_0$  alone. In particular, defining

$$\hat{\mathcal{D}}_0 \equiv e^{\frac{\lambda}{2} P_0} \tilde{P}_0, \tag{3.28}$$

the action of the proper time-translation generator  $\hat{\mathcal{D}}_0$  on the currents obtained from the Noether analysis vanishes:

$$\hat{\mathcal{D}}_0 J_{0A} = 0. \tag{3.29}$$

The fact that the Lagrangian density variation occurs exactly in the form (3.25)



is thus a relief if one is looking for an analogous of the 4-divergence of the currents in the Noether analysis of ordinary theories in classical Minkowski spacetime. To make this concept clearer, let us introduce the following rule of spatial integration in  $\kappa$ -Minkowski:

$$\int d^3x e^{ip \cdot \hat{x}} e^{-ip_0 \hat{x}_0} = \delta(\vec{p}) e^{-ip_0 \hat{x}_0}, \quad (3.30)$$

so that for a  $\kappa$ -Minkowski field  $\Psi(\hat{x}) = \int d^4p \tilde{\Psi}(p_0, \vec{p}) \exp(ip \cdot \hat{x}) \exp(-ip_0 \hat{x}_0)$  one obtains

$$\int d^3x \Psi(\hat{x}_0, \hat{x}) = \int dp_0 \tilde{\Psi}(p_0, \vec{0}) e^{-ip_0 \hat{x}_0}. \quad (3.31)$$

This spatial integration rule together with the action of the operator  $\hat{\mathcal{D}}_0$  (3.29) allows to write

$$\hat{\mathcal{D}}_0 \int d^3x J_{0A} = \int d^3x \hat{\mathcal{D}}_0 J_{0A} = - \int d^3x \hat{P}^i J_{iA} = 0, \quad (3.32)$$

from which the time independence of the charges  $\int d^3x J_{0A}$  follows from a generalization of the classical 4-divergence. We want to stress how all this argument holds without the introduction of the Weyl map and, thereby, without any reference to the classical case. In particular, we do not apply the technique (Gauss theorem) valid in classical Minkowski, that allows to transform the last integral of (3.32) into an integral over a surface where the fields vanish. The vanishing of  $\int d^3x \hat{P}^i J_{iA}$  follows directly from the action of the operator  $\hat{P}^i$  on a product of functions of  $\kappa$ -Minkowski, codified in the structure of the coproduct (3.13),

$$\begin{aligned} \hat{P}_i \left( e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0} \cdot e^{i\vec{p} \cdot \vec{x}} e^{-ip_0 x_0} \right) &= \hat{P}_i \left( e^{i(\vec{k} + e^{-\lambda k_0} \vec{p}) \cdot \vec{x}} e^{-i(k_0 + p_0) x_0} \right) = \\ &= (k_i + e^{-\lambda k_0} p_i) e^{\lambda(k_0 + p_0)} \left( e^{i(\vec{k} + e^{-\lambda k_0} \vec{p}) \cdot \vec{x}} e^{-i(k_0 + p_0) x_0} \right) = \\ &= (k_i e^{\lambda(k_0 + p_0)} + e^{\lambda p_0} p_i) \left( e^{i(\vec{k} + e^{-\lambda k_0} \vec{p}) \cdot \vec{x}} e^{-i(k_0 + p_0) x_0} \right) = \\ &= \hat{P}_i \left( e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0} \right) \cdot e^{\lambda p_0} \left( e^{i\vec{p} \cdot \vec{x}} e^{-ip_0 x_0} \right) + \left( e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0} \right) \cdot \hat{P}_i \left( e^{i\vec{p} \cdot \vec{x}} e^{-ip_0 x_0} \right) \end{aligned}$$

and the appearance of  $\delta(k_i + e^{-\lambda k_0} p_i)$ , when expanding the fields  $\Phi(\hat{x})$  over the time-to-the-right-ordered plane waves, as a consequence of the integration rule (3.30).

However, the conservation of the charges  $\int d^3x J_{0A}$  will be directly verified in the next section, where we explicitly compute them.

### 3.2.3 Conserved charges

We are now ready to derive the charges, that must be evaluated on the solutions of the equation of motion whose general form is given in (2.8). To show that they are time-independent we proceed analyzing separately  $J_{00}$ ,  $J_{0i}$  and  $J_{04}$ . For  $J_{00}$  we have:

$$\begin{aligned}
\hat{Q}_0 &= \int d^3x J_{00} = \frac{1}{2} \int d^3x \left\{ \left( \lambda \hat{P}_0 + e^{-\lambda P_0} \right) \cdot \right. \\
&\quad \cdot \left[ \phi \left( \frac{1 - e^{-\lambda P_0}}{\lambda} \right) - \left( \frac{e^{\lambda P_0} - 1}{\lambda} \right) \phi + \lambda m^2 \phi \right] \hat{P}_0 \phi + \\
&\quad + \lambda \hat{P}_i \left[ \phi \left( \frac{1 - e^{-\lambda P_0}}{\lambda} \right) - \left( \frac{e^{\lambda P_0} - 1}{\lambda} \right) \phi + \lambda m^2 \phi \right] \hat{P}_i \phi + \\
&\quad + \lambda \hat{P}_0 \left[ \phi \left( \frac{1 - e^{-\lambda P_0}}{\lambda} \right) - \left( \frac{e^{\lambda P_0} - 1}{\lambda} \right) \phi + \lambda m^2 \phi \right] \hat{P}_4 \phi \Big\} = \\
&= \frac{1}{2} \int d^3x d^4k d^4p \tilde{\Phi}(k_0, k_i) \tilde{\Phi}(p_0, p_i) e^{i(\vec{k} + e^{-\lambda k_0} \vec{p}) \cdot \vec{x}} e^{-i(k_0 + p_0) x_0} \cdot \\
&\quad \cdot \delta(C_\lambda(k) - m^2) \delta(C_\lambda(p) - m^2) \cdot \\
&\quad \cdot \left\{ \left( \lambda \hat{k}_0 + e^{-\lambda k_0} \right) \left( \frac{2 - e^{-\lambda p_0} - e^{\lambda k_0}}{\lambda} + \lambda m^2 \right) \hat{p}_0 + \right. \\
&\quad + \lambda k_i \left( \frac{2 - e^{-\lambda p_0} - e^{\lambda k_0}}{\lambda} + \lambda m^2 \right) \hat{p}_i + \lambda \hat{k}_0 \left( \frac{2 - e^{-\lambda p_0} - e^{\lambda k_0}}{\lambda} + \lambda m^2 \right) \hat{p}_4 \Big\} = \\
&= \frac{1}{2} \int d^4k d^4p \tilde{\Phi}(k_0, k_i) \tilde{\Phi}(p_0, p_i) e^{-i(k_0 + p_0) x_0} e^{3\lambda k_0} \cdot \\
&\quad \cdot \delta(\vec{p} + \vec{k} e^{\lambda k_0}) \delta(C_\lambda(k) - m^2) \delta(C_\lambda(p) - m^2) \cdot \\
&\quad \cdot \left\{ \left( \frac{2 - e^{-\lambda p_0} - e^{\lambda k_0}}{\lambda} + \lambda m^2 \right) \left[ \left( \lambda \hat{k}_0 + e^{-\lambda k_0} \right) \hat{p}_0 + \lambda k_i \hat{p}_i + \lambda \hat{k}_0 \hat{p}_4 \right] \right\} \\
&= \frac{1}{2} \int d^4k dp_0 \tilde{\Phi}(k_0, k_i) \tilde{\Phi}(p_0, -k_i e^{\lambda k_0}) e^{-i(k_0 + p_0) x_0} e^{3\lambda k_0} \cdot \\
&\quad \cdot \delta(\tilde{k}_0^2 - e^{\lambda k_0} k^2 - m^2) \delta(\tilde{p}_0^2 - e^{\lambda(p_0 + k_0)} \tilde{k}_0^2 + m^2 (e^{\lambda(p_0 + k_0)} - 1)) \cdot \\
&\quad \cdot \left\{ \left( \frac{2 - e^{-\lambda p_0} - e^{\lambda k_0}}{\lambda} + \lambda m^2 \right) \left[ \lambda \hat{k}_0 (\hat{p}_0 + \hat{p}_4) + \right. \right. \\
&\quad + \left. \left. e^{-\lambda k_0} \left( \frac{e^{\lambda p_0} - 1}{\lambda} - \frac{\lambda m^2}{2} \right) - \lambda k_i^2 e^{\lambda(p_0 + k_0)} \right] \right\}, \tag{3.33}
\end{aligned}$$

where  $\tilde{k}_\mu$  and  $\hat{k}_A$  are functions of the Fourier parameters  $k_\alpha$  of the same form as, respectively,  $\tilde{P}_\mu$  and  $\hat{P}_A$ , explicitly

$$\{\tilde{k}_0, \vec{k}\}_{|_{k_0, \vec{k}}} \equiv \left\{ \frac{2}{\lambda} \sinh\left(\frac{\lambda}{2} k_0\right), \vec{k} e^{\frac{\lambda}{2} k_0} \right\} \quad (3.34)$$

$$\{\hat{k}_0, \vec{k}, \hat{k}_4\}_{|_{k_0, \vec{k}}} \equiv \left\{ \frac{1}{\lambda} (\sinh \lambda k_0 + \frac{\lambda^2}{2} \vec{k}^2 e^{\lambda k_0}), \vec{k} e^{\lambda k_0}, \frac{1}{\lambda} (\cosh \lambda k_0 - 1 - \frac{\lambda^2}{2} \vec{k}^2 e^{\lambda k_0}) \right\}, \quad (3.35)$$

and we used the relation

$$\hat{P}_0 = \frac{e^{\lambda P_0} - 1}{\lambda} - \frac{\lambda m^2}{2}. \quad (3.36)$$

Looking at the requirement enforced by the second delta function

$$\tilde{p}_0^2 - e^{\lambda(p_0+k_0)} \tilde{k}_0^2 + m^2(e^{\lambda(p_0+k_0)} - 1) = 0,$$

one notices that it leads to two possible solutions

$$p_0^{(1)} = -k_0 \quad e^{-\lambda p_0^{(2)}} = 2 - e^{\lambda k_0} + \lambda^2 m^2.$$

On the first solution the  $\hat{Q}_0$  functional result to be time-independent, while on the second solution the time independence appears because of the vanishing of the  $\hat{Q}_0$  functional. In fact, the presence of the term  $\left(\frac{2-e^{-\lambda p_0}-e^{\lambda k_0}}{\lambda} + \lambda m^2\right)$  inside the expression of  $\hat{Q}_0$  gives straightforwardly a vanishing charge on the second solution.

Substituting the first solution of the second delta function,  $p_0 = -k_0$ , the value of the time-independent  $\hat{Q}_0$  functional will be given by

$$\begin{aligned} \hat{Q}_0 &= \frac{1}{2} \int d^4 k dp_0 \Phi(k) \Phi(p_0, \vec{k}) e^{3\lambda k_0} e^{-i(k_0+p_0)t} \delta(\tilde{k}_0^2 - e^{\lambda k_0} k^2 - m^2) \cdot \\ &\cdot \left\{ \left( \frac{2 - e^{-\lambda p_0} - e^{\lambda k_0}}{\lambda} + \lambda m^2 \right) \cdot \right. \\ &\cdot \left[ \lambda \hat{k}_0 (\hat{p}_0 + \hat{p}_4) + e^{-\lambda k_0} \left( \frac{e^{\lambda p_0} - 1}{\lambda} - \frac{\lambda m^2}{2} \right) - \lambda k_i^2 e^{\lambda(p_0+k_0)} \right] \Big\} \cdot \\ &\cdot \frac{\delta(k_0 + p_0)}{|\partial_{p_0} [\tilde{p}_0^2 - e^{\lambda(p_0+k_0)} \tilde{k}_0^2 + m^2(e^{\lambda(p_0+k_0)} - 1)]_{p_0=-k_0}|} \end{aligned}$$

$$\Rightarrow \hat{Q}_0 = -\frac{1}{2} \int d^4k \tilde{\Phi}(k) \tilde{\Phi}(\dot{-}k) e^{3\lambda k_0} \delta(C_\lambda(k) - m^2) \frac{(-2\tilde{k}_0 e^{\frac{\lambda}{2}k_0} + \lambda m^2)}{|-2\tilde{k}_0 e^{\frac{\lambda}{2}k_0} + \lambda m^2|} \hat{k}_0, \quad (3.37)$$

where we introduced the notations  $k \equiv (k_0, \vec{k})$ ,  $\dot{-}k \equiv (-k_0, -\vec{k}e^{\lambda k_0})$ .

The proof of the time independence of the  $\hat{Q}_i$  and  $\hat{Q}_4$  functionals will be similar to the one has been shown for the  $\hat{Q}_0$  functional. In particular

$$\begin{aligned} \hat{Q}_i &= \int d^3x J_{0i} = \frac{1}{2} \int d^3x \left\{ \lambda \hat{P}_i \left[ \phi \left( \frac{1 - e^{-\lambda P_0}}{\lambda} \right) - \left( \frac{e^{\lambda P_0} - 1}{\lambda} \right) \phi + \lambda m^2 \phi \right] \cdot \right. \\ &\quad \cdot \hat{P}_0 \phi + \left[ \phi \left( \frac{1 - e^{-\lambda P_0}}{\lambda} \right) - \left( \frac{e^{\lambda P_0} - 1}{\lambda} \right) \phi + \lambda m^2 \phi \right] \hat{P}_i \phi + \\ &\quad + \lambda \hat{P}_i \left[ \phi \left( \frac{1 - e^{-\lambda P_0}}{\lambda} \right) - \left( \frac{e^{\lambda P_0} - 1}{\lambda} \right) \phi + \lambda m^2 \phi \right] \hat{P}_4 \phi \Big\} = \\ &= \frac{1}{2} \int d^3x d^4k d^4p \tilde{\Phi}(k_0, k_i) \tilde{\Phi}(p_0, p_i) e^{i(\vec{k} + e^{-\lambda k_0} \vec{p}) \cdot \vec{x}} e^{-i(k_0 + p_0)x_0} \cdot \\ &\quad \cdot \delta(C_\lambda(k) - m^2) \delta(C_\lambda(p) - m^2) \left\{ \lambda \hat{k}_i \left( \frac{2 - e^{-\lambda p_0} - e^{\lambda k_0}}{\lambda} + \lambda m^2 \right) \hat{p}_0 + \right. \\ &\quad + \left( \frac{2 - e^{-\lambda p_0} - e^{\lambda k_0}}{\lambda} + \lambda m^2 \right) \hat{p}_i + \lambda \hat{k}_i \left( \frac{2 - e^{-\lambda p_0} - e^{\lambda k_0}}{\lambda} + \lambda m^2 \right) \hat{p}_4 \Big\} = \\ &= \frac{1}{2} \int d^4k d^4p \tilde{\Phi}(k_0, k_i) \tilde{\Phi}(p_0, p_i) e^{-i(k_0 + p_0)x_0} e^{3\lambda k_0} \cdot \\ &\quad \cdot \delta(\vec{p} + \vec{k} e^{\lambda k_0}) \delta(C_\lambda(k) - m^2) \delta(C_\lambda(p) - m^2) \cdot \\ &\quad \cdot \left\{ \left( \frac{2 - e^{-\lambda p_0} - e^{\lambda k_0}}{\lambda} + \lambda m^2 \right) \left[ \lambda \hat{k}_i \hat{p}_0 + \hat{p}_i + \lambda \hat{k}_i \hat{p}_4 \right] \right\} \\ &= \frac{1}{2} \int d^4k d^4p \tilde{\Phi}(k_0, k_i) \tilde{\Phi}(p_0, -k_i e^{\lambda k_0}) e^{-i(k_0 + p_0)x_0} e^{3\lambda k_0} \cdot \\ &\quad \cdot \delta(\tilde{k}_0^2 - e^{\lambda k_0} k^2 - m^2) \delta(\tilde{p}_0^2 - e^{\lambda(p_0 + k_0)} \tilde{k}_0^2 + m^2 (e^{\lambda(p_0 + k_0)} - 1)) \cdot \\ &\quad \cdot \left\{ \left( \frac{2 - e^{-\lambda p_0} - e^{\lambda k_0}}{\lambda} + \lambda m^2 \right) \left[ \lambda \hat{k}_i (\hat{p}_0 + \hat{p}_4) - k_i e^{\lambda(k_0 + p_0)} \right] \right\} \end{aligned} \quad (3.38)$$

and

$$\begin{aligned}
\hat{Q}_4 &= \int d^3x J_{04} = \frac{1}{2} \int d^3x \left\{ \left( \lambda \hat{P}_4 + 1 - e^{-\lambda P_0} \right) \cdot \right. \\
&\cdot \left[ \phi \left( \frac{1 - e^{-\lambda P_0}}{\lambda} \right) - \left( \frac{e^{\lambda P_0} - 1}{\lambda} \right) \phi + \lambda m^2 \phi \right] \hat{P}_0 \phi + \\
&- \lambda P_i \left[ \phi \left( \frac{1 - e^{-\lambda P_0}}{\lambda} \right) - \left( \frac{e^{\lambda P_0} - 1}{\lambda} \right) \phi + \lambda m^2 \phi \right] \hat{P}_i \phi + \\
&+ \left( \lambda \hat{P}_4 + 1 \right) \left[ \phi \left( \frac{1 - e^{-\lambda P_0}}{\lambda} \right) - \left( \frac{e^{\lambda P_0} - 1}{\lambda} \right) \phi + \lambda m^2 \phi \right] \hat{P}_4 \phi \Big\} = \\
&= \frac{1}{2} \int d^3x d^4k d^4p \tilde{\Phi}(k_0, k_i) \tilde{\Phi}(p_0, p_i) e^{i(\vec{k} + e^{-\lambda k_0} \vec{p}) \cdot \vec{x}} e^{-i(k_0 + p_0) x_0} \cdot \\
&\cdot \delta(C_\lambda(k) - m^2) \delta(C_\lambda(p) - m^2) \cdot \\
&\cdot \left\{ \left( \lambda \hat{k}_4 + 1 - e^{-\lambda k_0} \right) \left( \frac{2 - e^{-\lambda p_0} - e^{\lambda k_0}}{\lambda} + \lambda m^2 \right) \hat{p}_0 + \right. \\
&- \lambda k_i \left( \frac{2 - e^{-\lambda p_0} - e^{\lambda k_0}}{\lambda} + \lambda m^2 \right) \hat{p}_i + \\
&+ \left. \left( \lambda \hat{k}_4 + 1 \right) \left( \frac{2 - e^{-\lambda p_0} - e^{\lambda k_0}}{\lambda} + \lambda m^2 \right) \hat{p}_4 \right\} = \\
&= \frac{1}{2} \int d^4k d^4p \tilde{\Phi}(k_0, k_i) \tilde{\Phi}(p_0, p_i) e^{-i(k_0 + p_0) x_0} e^{3\lambda k_0} \cdot \\
&\cdot \delta(\vec{p} + \vec{k} e^{\lambda k_0}) \delta(C_\lambda(k) - m^2) \delta(C_\lambda(p) - m^2) \cdot \\
&\cdot \left\{ \left( \frac{2 - e^{-\lambda p_0} - e^{\lambda k_0}}{\lambda} + \lambda m^2 \right) \cdot \right. \\
&\cdot \left. \left[ \left( \lambda \hat{k}_4 + 1 - e^{-\lambda k_0} \right) \hat{p}_0 - \lambda k_i \hat{p}_i + \left( \lambda \hat{k}_4 + 1 \right) \hat{p}_4 \right] \right\} \\
&= \frac{1}{2} \int d^4k d^4p \tilde{\Phi}(k_0, k_i) \tilde{\Phi}(p_0, -k_i e^{\lambda k_0}) e^{-i(k_0 + p_0) x_0} e^{3\lambda k_0} \cdot \\
&\cdot \delta(\tilde{k}_0^2 - e^{\lambda k_0} k^2 - m^2) \delta(\tilde{p}_0^2 - e^{\lambda(p_0 + k_0)} \tilde{k}_0^2 + m^2 (e^{\lambda(p_0 + k_0)} - 1)) \cdot \\
&\cdot \left\{ \left( \frac{2 - e^{-\lambda p_0} - e^{\lambda k_0}}{\lambda} + \lambda m^2 \right) \cdot \right. \\
&\cdot \left. \left[ \left( \lambda \hat{k}_4 + 1 \right) (\hat{p}_0 + \hat{p}_4) - e^{-\lambda k_0} \left( \frac{e^{\lambda p_0} - 1}{\lambda} - \frac{\lambda m^2}{2} \right) + \lambda k_i^2 e^{\lambda(p_0 + k_0)} \right] \right\} .
\end{aligned} \tag{3.39}$$

It is now clear that both the relations (3.38) and (3.39) vanish on the solution  $e^{-\lambda k_0} = 2 - e^{\lambda p_0} + \lambda^2 m^2$ , while for the  $p_0 = -k_0$  solution the values of the time independent functionals are recovered:

$$\hat{Q}_i = -\frac{1}{2} \int d^4 k \tilde{\Phi}(k) \tilde{\Phi}(-k) e^{3\lambda k_0} \delta(C_\lambda(k) - m^2) \frac{(-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2)}{|-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2|} \hat{k}_i \quad (3.40)$$

and

$$\hat{Q}_4 = -\frac{1}{2} \int d^4 k \tilde{\Phi}(k) \tilde{\Phi}(-k) e^{3\lambda k_0} \delta(C_\lambda(k) - m^2) \frac{(-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2)}{|-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2|} \hat{k}_4. \quad (3.41)$$

Thus we can rewrite the charges in a more compact form

$$\begin{pmatrix} \hat{Q}_0 \\ \hat{Q}_i \\ \hat{Q}_4 \end{pmatrix} = -\frac{1}{2} \int d^4 k \Phi(k) \Phi(-k) e^{3\lambda k_0} \frac{(-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2)}{|-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2|} \begin{pmatrix} \hat{k}_0 \\ \hat{k}_i \\ \hat{k}_4 \end{pmatrix} \delta(C_\lambda(k) - m^2). \quad (3.42)$$

Thereby, our analysis leads to 5 translation-symmetry conserved charges from the 5D-calculus setup and, both in the massless and in the massive case, the charges we obtain are not classical: they are functional of the fields with a non-linear dependence on the Plank-scale  $\lambda$  and a delta of the deformed casimir. Just in the limit  $\lambda \rightarrow 0$  we reobtain the classical charges. Besides, our Noether analysis constructively led us to a “conservation equation” of the form  $\hat{\mathcal{D}}_0 J_A^0 + \hat{P}_i J_A^i = 0$ .

### 3.2.4 On a possible different choice of the 5D differential calculus basis

We have seen in section 3.2.2 that the proper time derivative operator is given by  $\hat{\mathcal{D}}_0 = \hat{P}_0 + \hat{P}_4$ . One may now look for a change of the differential calculus basis which enables us to rewrite the differential  $df$  in terms of the operator  $\hat{\mathcal{D}}_0$ , i.e. performing a change of basis for the transformation parameters such that the external derivative operator  $d$  still satisfies the Leibnitz rule. It is easy to see that the following change of basis (rotation) for the one-form generators<sup>4</sup>

$$\bar{d}\hat{x}_0 = (d\hat{x}_0 + d\hat{x}_4)/\sqrt{2} \quad \bar{d}\hat{x}_i = d\hat{x}_i \quad \bar{d}\hat{x}_4 = (d\hat{x}_0 - d\hat{x}_4)/\sqrt{2}, \quad (3.43)$$

endowed with the commutation relations

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<sup>4</sup>This change of basis was introduced by Sitarz [22] just for a reason of convenience in the presentation of the 5D differential calculus, without any physical intent.

$$\begin{aligned}
[\hat{x}_0, \bar{d}\hat{x}_0] &= i\lambda\bar{d}\hat{x}_0, & [\hat{x}_0, \bar{d}\hat{x}_4] &= -i\lambda\bar{d}\hat{x}_4, & [\hat{x}_0, \bar{d}\hat{x}_j] &= 0, \\
[\hat{x}_j, \bar{d}\hat{x}_4] &= 0, & [\hat{x}_j, \bar{d}\hat{x}_0] &= -\sqrt{2}i\lambda\bar{d}\hat{x}_j, & [\hat{x}_j, \bar{d}\hat{x}_k] &= -\sqrt{2}i\lambda\delta_{jk}\bar{d}\hat{x}_4,
\end{aligned} \tag{3.44}$$

suggest a natural way to write the differential  $df$ , in particular

$$df = (\bar{d}\hat{x}^0\bar{\mathcal{D}}_0 + \bar{d}\hat{x}^i\bar{\mathcal{D}}_i + \bar{d}\hat{x}^4\bar{\mathcal{D}}_4) f \equiv \bar{d}f, \tag{3.45}$$

where

$$\bar{\mathcal{D}}_0 = (\hat{P}_0 + \hat{P}_4)/\sqrt{2} = \hat{\mathcal{D}}_0/\sqrt{2} \quad \bar{\mathcal{D}}_i \equiv \hat{P}_i \quad \bar{\mathcal{D}}_4 = (\hat{P}_0 - \hat{P}_4)/\sqrt{2}. \tag{3.46}$$

The reason here to perform all the previous Noether analysis with this new basis for the transformation parameters (and hence for the translation generators), is to look for a more constraining characterization of the energy observable. The introduction of the proper time derivative operator from the beginning of the analysis, i.e. inside the definition of the differential of a generic  $\kappa$ -Minkowski element, might lead to a stronger intuition to identify a plausible energy charge<sup>5</sup>.

Following all the steps of section 3.2.2, one arrives to the expression for the Lagrangian density variation

$$\delta\mathcal{L} = \bar{d}\hat{x}^A (\bar{\mathcal{D}}^0 \bar{J}_{0A} + \bar{\mathcal{D}}^i \bar{J}_{iA}), \tag{3.47}$$

where

$$\bar{J}_{00} \equiv J_{00} + J_{04}, \quad \bar{J}_{0j} \equiv \sqrt{2} J_{0j}, \quad \bar{J}_{04} \equiv J_{00} - J_{04}, \tag{3.48}$$

$$\bar{J}_{i0} \equiv (J_{i0} + J_{i4})/\sqrt{2}, \quad \bar{J}_{ij} \equiv J_{ij}, \quad \bar{J}_{i4} \equiv (J_{i0} - J_{i4})/\sqrt{2}. \tag{3.49}$$

Thereby, the translation symmetry charges obtained from the rotation of the

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<sup>5</sup>Even though we started from the request that the time derivative operator  $\hat{\mathcal{D}}_0$  enter the expression of the differential  $df$ , arriving in this way to the one-form generators change of basis (3.43) satisfying this request, it can be shown that working within this new basis for the differential calculus one straightforwardly obtains the new basis for the translation generators  $\{\bar{\mathcal{D}}_0, \bar{\mathcal{D}}_i, \bar{\mathcal{D}}_4\}$ . A less rigorous but more physically intuitive procedure might consist of a direct manipulation of the  $\delta\mathcal{L}$  expression (3.25). For example, led by the intuition that the charge associated to the current  $J_{00} + J_{04}$  could represent a good candidate for the energy observable, one could notice that  $\hat{e}^A \hat{\mathcal{D}}^0 J_{0A} = \hat{\mathcal{D}}^0 (d\hat{x}^0 (J_{00} + J_{04}) + d\hat{x}^i J_{0i} + (d\hat{x}^4 - d\hat{x}^0) J_{04})$ .

$d\hat{x}_A$  basis (3.43) are

$$\begin{aligned}
\begin{pmatrix} \bar{Q}_0 \\ \bar{Q}_i \\ \bar{Q}_4 \end{pmatrix} &= -\frac{1}{2} \int d^4k \left| \tilde{\Phi}(k) \right|^2 \begin{pmatrix} \hat{k}_0 + \hat{k}_4 \\ \sqrt{2} \hat{k}_i \\ \hat{k}_0 - \hat{k}_4 \end{pmatrix} \frac{(-2\tilde{k}_0 e^{\frac{\lambda}{2}k_0} + \lambda m^2)}{|-2\tilde{k}_0 e^{\frac{\lambda}{2}k_0} + \lambda m^2|} \delta(C_\lambda(k) - m^2) = \\
&= \begin{pmatrix} \hat{Q}_0 + \hat{Q}_4 \\ \sqrt{2} \hat{Q}_i \\ \hat{Q}_0 - \hat{Q}_4 \end{pmatrix}.
\end{aligned} \tag{3.50}$$

We will see in the next Chapter how  $\bar{Q}_0$  might turn out to be a valuable tool, since it is the conserved charge associated with the transformation parameter  $\bar{d}\hat{x}_0$ , and therefore (in light of the fact that in  $\bar{d}f$  we have  $\bar{d}\hat{x}_0$  multiplying  $\bar{\mathcal{D}}_0$ , which is a plausible time-translation generator) is a plausible candidate for the energy charge.





## Chapter 4

# Energy-momentum dispersion relation

The interpretation that Quantum Group language gives to “momenta” as generators of translations (i.e. the real physical particle momenta) is based on the notion of quantum group symmetry. Chapters 2 and 3 provide two examples of the freedom there exist in the description of  $\kappa$ -Minkowski symmetries by anyone of a large number of basis of the  $\kappa$ -Poincaré Hopf algebra. The nature of this symmetry-description degeneracy remains obscure from a physics perspective, in particular we are used to associate energy-momentum with the translation generators and it is not conceivable that a given operative definition of energy-momentum could be equivalently described in terms of different translation generators. The difference would be easily established by testing, for example, the different dispersion relations (a meaningful physical property) that the different momenta satisfy.

In this context the claim for a Plank-scale modification of the energy-momentum relation is a crucial key-point. We saw in (2.24) how, using a four-dimensional differential calculus and the Majid-Ruegg  $\kappa$ -Poincaré basis for the translation generators, a non-linear Plank-scale modification of the dispersion relation for a free massless scalar field can be obtained. In this chapter we will look for a possible modification of the energy-momentum relation using the charges we obtained with the five-dimensional bicovariant differential calculus (3.7).

Since the 5D differential calculus is bicovariant under the action of the full  $\kappa$ -Poincaré algebra and the basis generators  $\hat{P}_0, \hat{P}_i$  of translations transform under  $\kappa$ -Poincaré action in the same way as the operators  $P_\mu$  in the commutative case transform under the standard Poincaré action, it could be expected that the energy-momentum relation remains classical. But this aspect of the 5D differential calculus should not mislead the analysis of  $\kappa$ -Minkowski translational symmetry, in fact the linearity of the  $\kappa$ -Poincaré action on the commutation relation (3.7) induces a highly non-trivial structure in the coalgebra sector of

the generators  $\hat{P}_A$  and thereby a non-trivial modification of the quantum symmetry.

Recovering a special-relativistic dispersion relation at the end of the analysis would seem less likely than expected and we will see that this indeed does not happen when considering a massive scalar field.

#### 4.1 Dispersion relation for regularized plane-wave field $\Phi \in \mathbb{C}$

The expressions for the charges obtained in (3.42) can now be used to investigate if there is any Plank-scale modification of the energy-momentum relation with respect to the special-relativistic (Poincaré-Lie-algebra) limit. We intend to probe the structure of the dispersion relation by using a “regularized plane-wave” field. In preparation for that we first rewrite the charges (3.42) in a more compact form.

For a real scalar classic field  $\Phi$ , solution of  $C_\lambda(k)\Phi = m^2\Phi$  on  $\kappa$ -Minkowski

$$\Phi(x) = \int d^4k \tilde{\Phi}(k) \delta(C_\lambda(k) - m^2) e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0}$$

holds the reality condition

$$\tilde{\Phi}(k_0, \vec{k}) = \left( \tilde{\Phi}(-k_0, -\vec{k} e^{\lambda k_0}) \right)^* e^{3\lambda k_0} \quad (4.1)$$

that allows us to rewrite the charges as

$$\begin{pmatrix} \hat{Q}_0 \\ \hat{Q}_i \\ \hat{Q}_4 \end{pmatrix} = -\frac{1}{2} \int d^4k |\tilde{\Phi}(k_0, \vec{k})|^2 \frac{(-2\tilde{k}_0 e^{\frac{\lambda}{2}k_0} + \lambda m^2)}{|-2\tilde{k}_0 e^{\frac{\lambda}{2}k_0} + \lambda m^2|} \begin{pmatrix} \hat{k}_0 \\ \hat{k}_i \\ \hat{k}_4 \end{pmatrix} \delta(C_\lambda(k) - m^2). \quad (4.2)$$

We now want to demonstrate that eq. (4.1) holds for complex fields too, i.e.

$$\tilde{\Phi}(k_0, \vec{k}) = \left( \tilde{\Phi}^*(-k_0, -\vec{k} e^{\lambda k_0}) \right)^* e^{3\lambda k_0} \quad (4.3)$$

In fact

$$\begin{aligned}
\Phi^*(x) &= \int d^4k \tilde{\Phi}^*(k) \delta(C_\lambda(k) - m^2) e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0} = \\
&= \int d^4k' e^{-3\lambda k'_0} \tilde{\Phi}^*(k'_0, \vec{k}' e^{-\lambda k'_0}) \delta(C_\lambda(k'_0, \vec{k}' e^{-\lambda k'_0}) - m^2) e^{i\vec{k}' e^{-\lambda k'_0} \cdot \vec{x}} e^{-ik'_0 x_0} = \\
&= \int d^4k e^{3\lambda k_0} \tilde{\Phi}^*(-k_0, -\vec{k} e^{\lambda k_0}) \delta(C_\lambda(-k_0, -\vec{k} e^{\lambda k_0}) - m^2) e^{-i\vec{k} e^{\lambda k_0} \cdot \vec{x}} e^{+ik_0 x_0} = \\
&= \int d^4k e^{3\lambda k_0} \tilde{\Phi}^*(-k_0, -\vec{k} e^{\lambda k_0}) \delta(C_\lambda(k_0, \vec{k}) - m^2) \left( e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0} \right)^* ; \quad (4.4)
\end{aligned}$$

conjugating now the last term of (4.1)

$$\Phi(x) = \int d^4k \left( e^{3\lambda k_0} \tilde{\Phi}^*(-k_0, -\vec{k} e^{\lambda k_0}) \right)^* \delta(C_\lambda(k_0, \vec{k}) - m^2) e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0} \quad (4.5)$$

and comparing (4.5) with

$$\Phi(x) = \int d^4k \tilde{\Phi}(k) \delta(C_\lambda(k) - m^2) e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0} ,$$

one immediately has eq. (4.3), which can be rewritten as

$$e^{-3\lambda k_0} \left( \tilde{\Phi}(k_0, \vec{k}) \right)^* = \tilde{\Phi}^*(-k_0, -\vec{k} e^{\lambda k_0}) . \quad (4.6)$$

We want to compute the translation-symmetry charges for a complex scalar field in order to compare our results with those of [27, 28]. In the previous chapter we considered real scalar fields, but actually the steps of the analysis are very similar for the case of complex fields. Essentially it reduces to the fact that in appropriate places one must consider the complex conjugate  $\Phi^*(x)$  of the field  $\Phi(x)$ . The action to use for a complex field is

$$\begin{aligned}
S[\Phi] &= \int d^4x \mathcal{L}[\Phi(x)] \\
\mathcal{L}[\Phi(x)] &= (\Phi^*(x) C_\lambda \Phi(x) - m^2 \Phi^*(x) \Phi(x)) , \quad (4.7)
\end{aligned}$$

and proceeding exactly in the same way as in the previous chapter one then easily arrives once again to the equation  $d\hat{x}^A \left( e^{\frac{\lambda P_0}{2}} \tilde{P}^0 J_{0A} + \hat{P}^i J_{iA} \right) = 0$ , with  $J$ 's of the same form as in the previous section but involving  $\Phi^*(x)$  in appropriate places. In particular, one finds

$$\begin{aligned}
J_{00} &= \left\{ \left( \frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \left[ (\lambda \hat{P}_0 + e^{-\lambda P_0}) \Phi^* \hat{P}_0 \Phi + \lambda P_i \Phi^* \hat{P}_i \Phi + \lambda \hat{P}_0 \Phi^* \hat{P}_4 \Phi \right] + \right. \\
&\quad \left. - (\lambda \hat{P}_0 + e^{-\lambda P_0}) \Phi^* \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_0 \Phi - \lambda P_i \Phi^* \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_i \Phi - \lambda \hat{P}_0 \Phi^* \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_4 \Phi \right\}, \\
J_{0i} &= \left\{ \left( \frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \left[ \lambda \hat{P}_i \Phi^* \hat{P}_0 \Phi + \Phi^* \hat{P}_i \Phi + \lambda \hat{P}_i \Phi^* \hat{P}_4 \Phi \right] + \right. \\
&\quad \left. - \lambda \hat{P}_i \frac{e^{-\lambda P_0}}{\lambda} \Phi^* \hat{P}_0 \Phi - \Phi^* \hat{P}_i \frac{e^{-\lambda P_0}}{\lambda} \Phi - \lambda \hat{P}_i \Phi^* \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_4 \Phi \right\}, \\
J_{04} &= \left\{ \left( \frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \left[ (\lambda \hat{P}_4 + 1 - e^{-\lambda P_0}) \Phi^* \hat{P}_0 \Phi - \lambda P_i \Phi^* \hat{P}_i \Phi + \right. \right. \\
&\quad \left. + (\lambda \hat{P}_4 + 1) \Phi^* \hat{P}_4 \Phi \right] - (\lambda \hat{P}_4 + 1 - e^{-\lambda P_0}) \Phi^* \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_0 \Phi + \\
&\quad \left. + \lambda P_i \Phi^* \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_i \Phi - (\lambda \hat{P}_4 + 1) \Phi^* \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_4 \Phi \right\}. \tag{4.8}
\end{aligned}$$

Hence the Noether analysis reported in Chapter 3 straightforwardly gives, for a complex scalar field, the following expression for the charges:

$$\begin{pmatrix} \hat{Q}_0 \\ \hat{Q}_i \\ \hat{Q}_4 \end{pmatrix} = - \int d^4 k \Phi(k) \Phi^*(\dot{-}k) e^{3\lambda k_0} \frac{(-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2)}{|-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2|} \begin{pmatrix} \hat{k}_0 \\ \hat{k}_i \\ \hat{k}_4 \end{pmatrix} \delta(C_\lambda(k) - m^2). \tag{4.9}$$

Eq. (4.6) enables us to rewrite the charges for a complex scalar field in the more compact form

$$\begin{pmatrix} \hat{Q}_0 \\ \hat{Q}_i \\ \hat{Q}_4 \end{pmatrix} = - \int d^4 k |\tilde{\Phi}(k_0, \vec{k})|^2 \frac{(-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2)}{|-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2|} \begin{pmatrix} \hat{k}_0 \\ \hat{k}_i \\ \hat{k}_4 \end{pmatrix} \delta(C_\lambda(k) - m^2) \tag{4.10}$$

And therefore it is clear that also for complex fields the translation-symmetry charges are real.

To write the plane-wave field  $\Phi_0^{p.w.}(x)$  solution of the deformed Klein-Gordon equation we need to calculate first the solutions of  $\delta(C_\lambda(k) - m^2)$ :

$$\delta(C_\lambda(k) - m^2) = \delta \left( \left( \frac{2}{\lambda} \sinh \frac{\lambda k_0}{2} \right)^2 - |\vec{k}|^2 e^{\lambda k_0} - m^2 \right) =$$

$$= \frac{1}{2\sqrt{m^2 + |\vec{k}|^2 + \lambda^2 m^4/4}} (\delta(k_0 - k_0^+) + \delta(k_0 - k_0^-)), \quad (4.11)$$

where

$$k_0^+ = \frac{1}{\lambda} \ln \left( \frac{1 + (\lambda m)^2/2 + \lambda \sqrt{m^2 + |\vec{k}|^2 + \lambda^2 m^4/4}}{1 - (\lambda |\vec{k}|)^2} \right)$$

$$k_0^- = \frac{1}{\lambda} \ln \left( \frac{1 + (\lambda m)^2/2 - \lambda \sqrt{m^2 + |\vec{k}|^2 + \lambda^2 m^4/4}}{1 - (\lambda |\vec{k}|)^2} \right); \quad (4.12)$$

from the signs analysis it's easy to see that  $k_0^+$  is positive and  $k_0^-$  is negative in the definition dominion  $|\vec{k}| < \frac{1}{\lambda}$ . It can also be seen that  $k_0^+$  is real only in the dominion  $|\vec{k}| < \frac{1}{\lambda}$ . Besides, it's obvious how in the "classic limit"  $\lambda \rightarrow 0$ ,  $k_0^+$  and  $k_0^-$  go respectively to the positive,  $\sqrt{|\vec{k}|^2 + m^2}$ , and negative,  $-\sqrt{|\vec{k}|^2 + m^2}$ , "classic" frequencies.

Setting  $N = 2\sqrt{m^2 + |\vec{k}|^2 + \lambda^2 m^4/4}$ , the particular regularized plane-wave  $\Phi_0^{p.w.}(x)$ , solution of the equation of motion, can be written as:

$$\begin{aligned} \Phi_0^{p.w.}(x) &= \int d^4k \frac{\sqrt{N}\theta(k_0)\delta(\vec{k} - \vec{p})}{\sqrt{V}} e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0} \delta(C_\lambda(k_0, \vec{k}) - m^2) = \\ &= \int \frac{d^4k}{N} \frac{\sqrt{N}\delta(\vec{k} - \vec{p})}{\sqrt{V}} e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0} \delta(k_0 - k_0^+) = \\ &= \frac{1}{(2V\sqrt{m^2 + |\vec{p}|^2 + \lambda^2 m^4/4})^{\frac{1}{2}}} e^{i\vec{p} \cdot \vec{x}} e^{-ip_0^+ x_0}, \end{aligned} \quad (4.13)$$

where  $V$  is a normalization spatial volume of the plane-wave.

Noting that  $(-2\tilde{k}_0 e^{\frac{\lambda}{2}k_0} + \lambda m^2)$  is negative for  $|\vec{k}| < \frac{1}{\lambda}$ , we are now ready to compute the charges:

$$\begin{aligned}
\begin{pmatrix} \hat{Q}_0^{p.w.} \\ \hat{Q}_i^{p.w.} \\ \hat{Q}_4^{p.w.} \end{pmatrix} &= - \int d^4 k \begin{pmatrix} \hat{k}_0 \\ \hat{k}_i \\ \hat{k}_4 \end{pmatrix} \left| \frac{\sqrt{N} \theta(k_0) \delta(\vec{k} - \vec{p})}{\sqrt{V}} \right|^2 \frac{(-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2)}{|-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2|} \delta(C_\lambda(k) - m^2) = \\
&= - \int d^4 k \begin{pmatrix} \hat{k}_0 \\ \hat{k}_i \\ \hat{k}_4 \end{pmatrix} \left| \sqrt{N} \right|^2 \frac{(-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2)}{|-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2|} \theta(k_0) \delta(\vec{k} - \vec{p}) \delta(C_\lambda(k) - m^2) = \\
&= \int d^4 k \begin{pmatrix} \hat{k}_0 \\ \hat{k}_i \\ \hat{k}_4 \end{pmatrix} \frac{|\sqrt{N}|^2}{N} \delta(k_0 - k_0^+) \delta(\vec{k} - \vec{p}) = \\
&= \int d^3 k \begin{pmatrix} \hat{k}_0(k_0^+, \vec{k}) \\ \hat{k}_i(k_0^+, \vec{k}) \\ \hat{k}_4(k_0^+, \vec{k}) \end{pmatrix} \delta(\vec{k} - \vec{p}) = \\
&= \begin{pmatrix} \hat{k}_0|_{k_0=p_0^+, \vec{k}=\vec{p}} \\ \hat{k}_i|_{k_0=p_0^+, \vec{k}=\vec{p}} \\ \hat{k}_4|_{k_0=p_0^+, \vec{k}=\vec{p}} \end{pmatrix} \tag{4.14}
\end{aligned}$$

“on shell” with respect to the deformed Casimir of the bicrossproduct basis  $C_\lambda(p_0^+, \vec{p})$ .

In light of the hypothesis of a modified dispersion relation for particles in the context of Quantum Groups approach to the problem of Quantum Gravity, we may now investigate this possibility. A key role in this investigation is played by the energy observable, i.e. the identification of a plausible candidate for the energy observable is a fundamental step for any claim.

It is perhaps intriguing that, from the equation of motion  $C_\lambda(p_0^+, \vec{p}) = m^2$ ,

$$\begin{aligned}
(\hat{Q}_0^{p.w.})^2 - (\hat{Q}_i^{p.w.})^2 &= \left( \frac{e^{\lambda p_0^+} - 1}{\lambda} - \frac{\lambda m^2}{2} \right)^2 - (\vec{p} e^{\lambda p_0^+})^2 = \\
&= m^2 \left( 1 + \frac{\lambda^2 m^2}{4} \right) = m^2 + (\hat{Q}_4^{p.w.})^2, \tag{4.15}
\end{aligned}$$

but this should be analyzed taking into consideration the fact that, in light of the observations we reported in section 3.2.2 on time translations,  $\hat{Q}_0^{p.w.}$  clearly cannot be the energy carried by our regularized plane wave.

The Noether analysis reported in section 3.2.4 for a rotated basis of the transformation parameters, might now be taken into consideration to contemplate a role for the combination  $\hat{Q}_0^{p.w.} + \hat{Q}_4^{p.w.}$ , which has emerged as the conserved charge associated with a transformation parameter ( $\bar{d}\hat{x}_0$ ) that can be

meaningfully described as a time-translation parameter. Thus, taking  $\hat{Q}_0^{p.w.} + \hat{Q}_4^{p.w.}$  as a candidate for the energy observable, we find that

$$\begin{aligned} (\hat{Q}_0^{p.w.} + \hat{Q}_4^{p.w.})^2 - (\hat{Q}_i^{p.w.})^2 &= \left( \frac{e^{\lambda p_0^+} - 1}{\lambda} \right)^2 - (\vec{p} e^{\lambda p_0^+})^2 = \\ &= \left( \lambda(\hat{Q}_0^{p.w.} + \hat{Q}_4^{p.w.}) + 1 \right) m^2. \end{aligned} \quad (4.16)$$

Eq. (4.16) shows how in the massless case there is no Plank-scale modification of the energy-momentum relation, the dispersion relation is classical, as in [27, 28], even though the charges are not. While, in the massive case, there is a  $\lambda$  deformation and the energy-momentum relation is no more special-relativistic, in fact the right-side term of (4.16) is not a relativistic invariant.

However, it is interesting to notice that, as in the analysis with a four-dimensional differential calculus reported in Chapter 2, this modification vanishes if one increases arbitrary the intensity of the fields. In fact, if we rescale the fields  $\Phi_0$  by a factor  $A$ , we have

$$\Phi_0^R = A\Phi_0 \quad \Rightarrow \quad (\hat{Q}_0^{p.w.R} + \hat{Q}_4^{p.w.R}, \hat{Q}_i^{p.w.R}) = A^2(\hat{Q}_0^{p.w.} + \hat{Q}_4^{p.w.}, \hat{Q}_i^{p.w.}) \quad (4.17)$$

and, rewriting eq. (4.16),

$$\begin{aligned} \frac{(\hat{Q}_0^{p.w.R} + \hat{Q}_4^{p.w.R})^2}{A^4} - \frac{(\hat{Q}_i^{p.w.R})^2}{A^4} &= m^2 \left( 1 + \lambda \frac{(\hat{Q}_0^{p.w.R} + \hat{Q}_4^{p.w.R})}{A^2} \right) \\ \Rightarrow \quad (\hat{Q}_0^{p.w.R} + \hat{Q}_4^{p.w.R})^2 - (\hat{Q}_i^{p.w.R})^2 &= (A^2 m)^2 \left( 1 + \lambda \frac{(\hat{Q}_0^{p.w.R} + \hat{Q}_4^{p.w.R})}{A^2} \right). \end{aligned} \quad (4.18)$$

So, in the limit  $A \rightarrow \infty$ , the special-relativistic relation is reestablished with mass  $m^R = A^2 m$ .





# Conclusions

In this thesis work we have investigated the connection between Hopf-algebra-type symmetry (Quantum symmetry) and five-dimensional bicovariant differential calculus, both concepts already present in the literature but never combined together within a rigorous and comprehensive analysis. The concept of Quantum symmetry in the context of  $\kappa$ -Minkowski noncommutative spacetime has revealed many interesting aspects but also some ambiguities. The formulation in terms of Hopf algebra (quantum) version of the classical Poincaré group for the symmetries of a free scalar field theory in  $\kappa$ -Minkowski provided a solid background for a Noether analysis of these symmetries. But the classical interpretation of real physical particle momenta as the conserved charges associated to the translation generators seems to be puzzling in the Quantum Group language. In fact, in Chapter 1 we illustrated the freedom there exist in the description of  $\kappa$ -Minkowski symmetries by anyone of a large number of  $\kappa$ -Poincaré Hopf algebra basis. In particular, we are left with a choice between different realization of the concept of translations in the noncommutative spacetime. This symmetry-description degeneracy raises a puzzling question: which translation generators basis gives the real physical energy-momentum charges?

Besides, the discovery of the central role that the introduction of a differential calculus has in order to be able to complete a Noether analysis brings about further ambiguities. We have seen in Chapters 2 and 3 that, while in the commutative case there is only one natural differential calculus, involving the conventional derivatives, in the  $\kappa$ -Minkowski case the introduction of a differential calculus is not unique. The construction of the  $\kappa$ -Minkowski spacetime enables us to use the tools of noncommutative geometry to construct  $\kappa$  deformations of field theory. The differential calculus, being the most important tool, is therefore a crucial point of these efforts. The search of a differential calculus that is left invariant under the action of the full  $\kappa$ -Poincaré algebra seems to be a reasonable choice and might have motivated some authors to think that the “classical case” might be recovered. The fact that the new translation generators basis  $\hat{P}_A$  introduced by the 5D differential calculus transform under  $\kappa$ -Poincaré action with the same commutation relations of the commutative case, i.e as the classical operators  $P_\mu$  on Minkowski spacetime, is quite sur-

prising and might motivate the attribution to these generators of a special or privileged role. In Chapter 3 we have shown that this sort of linearity of the operators  $\hat{P}_A$  has as counterpart an high non-trivial structure of the coalgebra sector and of the commutation relations of the one-form elements of the 5D differential calculus with the time-to-the-right-ordered plane wave basis of  $\kappa$ -Minkowski. Thereby the overall structure of the quantum symmetry is once again far from satisfactory. This thesis work shows that the elegant, and, from a certain point of view, natural requirement of bicovariance of the differential calculus adopted is not strong enough to eliminate the peculiarities of the new type of symmetry we have to deal with in the Quantum Groups scenario.

Nevertheless, some interesting properties of the 5D differential calculus are revealed once we use it to perform a Noether analysis of translation symmetries in  $\kappa$ -Minkowski. The fact that we have five  $d\hat{x}_A$  one-forms, and thus we expect five currents in the analysis, could represent a challenge for the physical interpretation, once we use these currents in order to look for conserved charges. The results of Chapter 3 show that the 5D-calculus-based translation transformations can indeed be implemented as symmetries of theories in  $\kappa$ -Minkowski. Our analysis performed directly within the noncommutative theory also allowed us to investigate explicitly the properties of the 5 “would-be currents”, dissolving all the initial worries and constructively leading us to current-conservation-like equations written in terms of the operator  $\hat{D}_0$  which, rather than  $\hat{P}_0$ , is a plausible candidate for the generator of time translations. The real physical interpretation problem concerns the possibility or not to properly call these charges the *energy-momentum charges*. In fact, even though the change of basis for the 5D differential calculus introduced in section 3.2.4 led us to identify the new parameter  $\tilde{d}\hat{x}_0$  as a time-translation parameter and the charge  $\tilde{Q}_0$  as a plausible candidate of time-translation-symmetry charge, several logical-consistency checks should be performed before any definite claim. More on this point can be found in [39], which also compares the analysis here reported in Chapter 3 with [27, 28].

A possible way out of this physical description ambiguity would be provided by testing experimentally the different dispersion relations (a meaningful physical property) that the different momenta satisfy. In Section 2.2 and Chapter 4 we used the general formulas for the translation-symmetry charges obtained with the 4D and 5D differential calculi to derive their form for complex plane wave fields and their dispersion relations. In both cases we saw a  $\lambda$  deformation of the special-relativistic form. However, when we rescale the fields by a factor  $A$  the  $\lambda$ -dependent correction becomes less and less important as  $A$  is increased, and actually in the  $A \rightarrow \infty$  limit the new effect disappears and the dispersion relation regain its special-relativistic form. Thereby, from a phenomenological perspective, the possibility to discriminate between the different choices of generators basis and differential calculi seems very unlikely, since for all practical purposes (all realistically-large classical-field configurations) the

associated new effects are quantitatively irrelevant.

In this context the construction of a Quantum Field Theory in  $\kappa$ -Minkowski might have an important role in clarifying the status of energy-momentum observables and in dissolving the ambiguity concerning the description of translations. It seems plausible that, while classical fields are essentially unaffected by the symmetry deformation, quantum particles in  $\kappa$ -Minkowski spacetime be affected by a significant modification of the dispersion relation. Perhaps the theory we considered does not have enough structure to give proper physical significance to energy-momentum, while a Quantum Field Theory analysis, following the approach here advocated, might lead to further constraints to the connotation of energy-momentum observables.



## Appendix A

# Bicrossproduct Hopf algebras

In order to provide the definition of “Bicrossproduct” Hopf algebra we need first to introduce the concepts of action and coaction of an algebra. An algebra can act on other structures. A left action of an algebra  $H$  over an algebra  $A$  is a linear map  $\alpha : H \otimes A \rightarrow A$  such that:

$$\begin{aligned}\alpha((h \cdot g) \otimes a) &= \alpha(h \otimes \alpha(g \otimes a)) \quad h, g \in H, a \in A \\ \alpha(h \otimes a) &= \epsilon(h)a\end{aligned}\tag{A.1}$$

We can use the short notation  $\alpha(h \otimes a) = h \triangleright a$ , so the (A.1) can be written as

$$(hg) \triangleright (a) = h \triangleright (g \triangleright a) \tag{A.2}$$

$$h \triangleright 1 = \epsilon(h)1. \tag{A.3}$$

Usually in physical applications, the request of *covariant action* is made in order that the action of a Hopf algebra preserves the structure of the object on which it acts. We say that an Hopf algebra  $H$  acts *covariantly* (from the left) over an algebra  $A$  (or equivalently that  $A$  is a left  $H$ -module algebra) if  $\forall h \in H$ :

$$h \triangleright (a \cdot b) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b) \quad a, b \in A.$$

The action of  $H$  over a coalgebra  $C$  ( $C$  is a left  $H$ -module coalgebra) states that:

$$\begin{aligned}\Delta(h \triangleright c) &= (h_{(1)} \triangleright c_{(1)}) \otimes (h_{(2)} \triangleright c_{(2)}) = (\Delta h) \triangleright \Delta c \\ \epsilon(h \triangleright c) &= \epsilon(h)\epsilon(c), \quad c \in C.\end{aligned}\tag{A.4}$$

In the same way it can be defined a right action of a Hopf algebra  $H$  on  $A$  (algebra, coalgebra or Hopf algebra):

$$\triangleleft : A \otimes H \rightarrow A$$

and a *covariant* right-action should satisfy

$$(a \cdot b) \triangleleft h = (a \triangleleft h_{(1)})(b \triangleleft h_{(2)}), \quad h \in H, \quad a, b \in A$$

The duality relations connect the left action over an algebra  $A$  and the corresponding right dual action over the dual algebra  $A^*$  in the following way:

$$\langle a, h \triangleright b \rangle = \langle a \triangleleft^* h, b \rangle, \quad b \in A, \quad a \in A^*, \quad h \in H. \quad (\text{A.5})$$

Let us make two examples of action, the adjoint action and the canonical action, and show that the latter reduces to the physical notion of translation in the case it is applied to the standard generators of the Poincaré-translations acting on its dual space (i.e. the commutative Minkowski space).

The left and right *adjoint actions* of a Hopf algebra  $H$  on itself are linear maps  $H \otimes H \rightarrow H$  such that

$$\begin{aligned} a \triangleright^{ad} b &= a_{(1)} b S(a_{(2)}), \\ b \triangleleft^{ad} a &= S(a_{(1)}) b a_{(2)}, \quad a, b \in H \end{aligned} \quad (\text{A.6})$$

These actions are covariant.

The left and right *canonical actions* of an algebra  $A$  over the dual coalgebra  $C \equiv A^*$  are defined as:

$$\begin{aligned} a \triangleright^{can} c &= c_{(1)} \langle a, c_{(2)} \rangle, \\ c \triangleleft^{can} a &= \langle c_{(1)}, a \rangle c_{(2)}, \quad a \in A, \quad c \in C \end{aligned} \quad (\text{A.7})$$

These actions are covariant as well.

The translation sector  $T$  of the Poincaré algebra is a Lie algebra (trivial Hopf algebra) generated by the operators  $P_\mu$  that has the following Hopf Algebra structure:

$$[P_\mu, P_\nu] = 0, \quad \Delta P_\mu = P_\mu \otimes 1 + 1 \otimes P_\mu, \quad \epsilon(P_\mu) = 0, \quad S(P_\mu) = 0. \quad (\text{A.8})$$

the duality axioms (1.19-1.23) allow us to reconstruct the Hopf-algebra structure of the Minkowski space  $M$  from the Hopf-algebra structure of its dual space  $T$ . In fact, assuming that the duality relations between the generators  $P_\mu$  of  $T$  and the generators  $x_\mu$  of its dual space  $M = T^*$  be<sup>1</sup>

$$\langle P_\mu, x_\nu \rangle = -i\eta_{\mu\nu}, \quad (\text{A.9})$$

one can easily find also  $M$  has a Lie-algebra structure

$$[x_\mu, x_\nu] = 0, \quad \Delta(x_\mu) = x_\mu \otimes 1 + 1 \otimes x_\mu, \quad S(x_\mu) = -x_\mu. \quad (\text{A.10})$$

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<sup>1</sup>We use a  $(+, -, -, -)$  signature.

Using these relations, the canonical action of  $P_\mu \in T$  on  $x_\mu \in T^*$  can be obtained:

$$P^\mu \triangleright^{can} x^\nu = x_{(1)}^\nu < P^\mu, x_{(2)}^\nu > = < P^\mu, x^\nu > = -i\eta_{\mu\nu} \quad (\text{A.11})$$

This is just the usual definition of the Poincaré translations in the case of commutative Minkowski spacetime in which the translation generators take the differential form  $P_\mu = -i\partial_\mu$  and its action over the coordinates is just  $P_\mu x_\nu = -i\eta_{\mu\nu}$ .

The dual concept to the action of an algebra is the *coaction* of a coalgebra. the left coaction of a coalgebra  $C$  over an algebra  $A$  is defined as a linear application  $\beta_L : A \rightarrow C \otimes A$ . The map  $\beta_L$  satisfies:

$$(id \otimes \Delta) \circ \beta_L = (\Delta \otimes id) \circ \beta_L \quad (\text{A.12})$$

$$(\epsilon \otimes id) \circ \beta_L = id \quad (\text{A.13})$$

The coaction gives a corepresentation of a coalgebra. A *covariant coaction* is required to respect the algebra structure on which it (co)acts. Thus:

$$\beta_L(ab) = \beta_L(a)\beta_L(b), \quad \beta_L(1) = 1 \otimes 1, \quad a, b \in A. \quad (\text{A.14})$$

We will adopt the following notation for the coaction:

$$\beta(a) = \sum_i a_i^{(\bar{1})} \otimes a_i^{(\bar{2})} = a^{(\bar{1})} \otimes a^{(\bar{2})} \quad a, a^{(\bar{2})} \in A, \quad a^{(\bar{1})} \in C. \quad (\text{A.15})$$

The notions of action and coaction allow us to define a special class of algebras that can be constructed by the composition of two Hopf algebras. These algebras are called *bicrossproduct algebras* and take an important role in our study since the  $\kappa$ -Poincaré algebra has been shown to be of this type [15]. Consider two Hopf algebras,  $A$  and  $X$ . Suppose that we know a right action  $\triangleleft$  of the algebra  $X$  over the algebra  $A$  and a left coaction of  $A$  on  $X$ :

$$\triangleleft : A \otimes X \rightarrow A, \quad (\text{A.16})$$

$$\beta_L : X \rightarrow A \otimes X. \quad (\text{A.17})$$

A bicrossproduct algebra (usually indicated with the symbol  $X \triangleright \triangleleft A$ ) is the tensor product algebra  $X \otimes A$  with the maps:

$$(x \otimes a) \cdot (y \otimes b) = xy_{(1)} \otimes (a \triangleleft y_{(2)})b \quad (\text{product}) \quad (\text{A.18})$$

$$1_{X \triangleright \triangleleft A} = 1_X \otimes 1_A \quad (\text{unity}) \quad (\text{A.19})$$

$$\Delta(x \otimes a) = [x_{(1)} \otimes x_{(2)}^{(\bar{1})} a_{(1)}] \otimes [x_{(2)}^{(\bar{2})} \otimes a_{(2)}] \quad (\text{coproduct}) \quad (\text{A.20})$$



$$\epsilon(x \otimes a) = \epsilon(x)\epsilon(a) \quad (\text{counit}) \quad (\text{A.21})$$

$$S(x \otimes a) = (1_X \otimes S(x^{(1)}a)) \cdot (S(x^{(2)}) \otimes 1_A) \quad (\text{antipode}) \quad (\text{A.22})$$

where  $a \in A$ ,  $x, y \in X$ .

One can consider as generators of the bicrossproduct algebra  $X \bowtie \triangleleft A$  the elements of the type  $A = 1 \otimes a$  and  $X = x \otimes 1$ . In fact, following the definition above, the single element  $x \otimes a \in X \bowtie \triangleleft A$  is given by the product  $XA$ :

$$XA = (x \otimes 1)(1 \otimes a) = x \otimes (1 \triangleleft 1)a = x \otimes a, \quad (\text{A.23})$$

while the other product  $AX$  is:

$$AX = (1 \otimes a)(x \otimes 1) = x_{(1)} \otimes (a \triangleleft x_{(2)}). \quad (\text{A.24})$$

Thus, the bicrossproduct algebra  $X \bowtie \triangleleft A$  can be viewed as the enveloping algebra generated by  $X$  and  $A$ , modulo the commutation relations:

$$[X, A] = x \otimes a - x_{(1)} \otimes (a \triangleleft x_{(2)}). \quad (\text{A.25})$$

The bicrossproduct  $\kappa$ -Poincaré algebra  $U(so(1, 3)) \bowtie \triangleleft T$  is constructed in this way, choosing  $X = U(so(1, 3))$ , the Lie algebra of Lorentz rotations, and  $A = T$ , the algebra of the Poincaré translations with a deformed coalgebra sector ( $T$  is a non-trivial Hopf algebra). In particular, in the Majid-Ruegg construction [15], the (deformed) coalgebra of  $T$  is chosen such that  $T$  is a dual space to  $\kappa$ -Minkowski, i.e. to the Lie algebra generated by the elements  $\hat{x}_\mu$  which satisfy the commutation relations

$$[\hat{x}_j, \hat{x}_0] = i\lambda\hat{x}_j, \quad [\hat{x}_j, \hat{x}_k] = 0. \quad (\text{A.26})$$

Quantum groups of bicrossproduct type were first proposed by S. Majid [15] as key ingredient for the unification of Gravity and Quantum Mechanics. This point of view is alternative to the idea of quantization of a theory as the result of a process applied to the underlying classical space (as for example in the case of deformation quantization, that is based on the classical notion of Poisson brackets or the case of String Theory description of quantum Gravity where one quantizes strings moving in a classical spacetime). Models should instead be built guided by the intrinsic Noncommutative Geometry at the level of noncommutative algebras. Only at the end one can consider classical geometry (with Poisson brackets) as classical limits and not as a starting point, like in *deformation quantization*. In a quantum world, in fact, phase-space and probably spacetime should be “fuzzy” and only approximately described by classical geometry.

The idea at the basis of the introduction of Quantum Groups in a Quantum Gravity approach is a principle of *self-duality*, that is peculiar in the Hopf

algebras and that should allow to put *quantum mechanics* and *gravity* on equal (but mutually dual) footing.

In order to explain this concept we make the example of the classical phase-space  $(q_j, p_j) \in R^{2n}, j = 1, \dots, n$ . In this case we can consider the group  $G$  of elements  $W_k = e^{ik_j q_j}$  labeled by the parameter  $k_j \in R^n$ . This group has an abelian composition law  $W_{k_1} W_{k_2} = W_{k_1+k_2}$ . The algebra of the functions of positions is given by the enveloping algebra  $U(g)$  (where  $g$  is the Lie algebra of  $G$ ). The algebra of the momenta can be viewed as the algebra  $C[G]$  dual to  $U(g)$ , and the generators  $p_j$  of this algebra can be introduced via the relations:

$$p_j(W_k) \equiv \langle p_j, W_k \rangle = k_j \quad (\text{A.27})$$

that follow from the duality relation  $\langle p_j, q_l \rangle = -i\delta_{jl}$ . Thus:

$$p_j(W_{k_1} W_{k_2}) = p_j(W_{k_1+k_2}) = (k_1 + k_2)_j. \quad (\text{A.28})$$

On the other hand (by duality),

$$\begin{aligned} p_j(W_{k_1} W_{k_2}) &= \langle p_j, W_{k_1} W_{k_2} \rangle = \langle \Delta(p_j), W_{k_1} \otimes W_{k_2} \rangle = \\ &= \Delta(p_j)(W_{k_1} \otimes W_{k_2}). \end{aligned} \quad (\text{A.29})$$

Thus the coproduct turns out to be:  $\Delta(p_j) = p_j \otimes 1 + 1 \otimes p_j$ . From this example one can see the relation between the coproduct and the composition law in the momentum sector. The commutativity of the positions is connected with the cocommutativity of coproduct in the space of momenta (then the momentum space is flat, *it has an abelian composition law*). If the space of positions is noncommutative, the group law of  $G$  will be in general non-abelian  $W_{k_1} W_{k_2} \neq W_{k_1+k_2}$  and the coproduct of momenta will be noncocommutative. Thus a noncommutative position space corresponds to a curved momentum space. The existence of *self-duality* however states also the opposite: a curved space in positions corresponds to a noncommutative momentum space.

In these terms,  $\kappa$ -Poincaré can have two different physical interpretations: one as quantum group symmetry, the other as quantized phase-space. In fact we have

$$P_k = U(so(1, 3)) \triangleright \triangleleft T = U(so(1, 3)) \triangleright \triangleleft \mathbb{C}(P), \quad (\text{A.30})$$

where on one side we have  $T$  as the deformed enveloping algebra of the Poincaré momenta sector. (A.30) underlines that the same construction can be described in terms of  $\mathbb{C}(P)$ , the algebra of functions on the “classical” but curved momenta space. Thus, our new phase-space is constructed by  $\kappa$ -Minkowski noncommutative spacetime and a curved momenta space. The two sectors are connected by a particular Fourier transform that we introduced in section (1.4)

The search for a quantum algebra of observable which is a Hopf algebra translates in a search of a simple model in which quantum and gravitational effects are unified and in which they are dual to each other.

Let us analyze more deeply the bicrossproduct structure of the Majid-Ruegg basis and let us show that it acts *covariantly* on  $\kappa$ -Minkowski. In the construction of a bicrossproduct algebra there are no prescriptions on the action of the elements of the bicrossproduct algebra itself. The choice of this action can be dictated by the generalization of the classical action of the standard Poincaré generators. In the case of the Poincaré algebra the action of the Lorentz rotations over the translation generators is represented by the commutators, and this action coincides with the adjoint action. Taking into account that the Poincaré algebra is generated by elements  $h$  that satisfy  $\Delta(h) = h \otimes 1 + 1 \otimes h$ ,  $S(h) = -h$ , one easily finds

$$p^j \triangleleft^{Ad} h = S(h_{(1)}) p^j h_{(2)} = [h, p^j]. \quad (\text{A.31})$$

This suggest to consider the adjoint action as a good generalization of the action of the Lorentz rotations and boosts over the translation generators, also in the deformed case. It is surprising that in the case of  $\kappa$ -Poincaré bicrossproduct basis the adjoint action is still given by the commutators. Taking into account the definition (A.6) and the  $\kappa$ -Poincaré structures in the bicrossproduct basis (1.32)-(1.33), one finds:

$$P^\mu \triangleleft^{Ad} N^j = S(N_{(1)}^j) P^\mu N_{(2)}^j = [P^\mu, N^j] \quad (\text{A.32})$$

$$P^\mu \triangleleft^{Ad} M^j = S(M_{(1)}^j) P^\mu M_{(2)}^j = [P^\mu, M^j]. \quad (\text{A.33})$$

Assuming that  $M_j, N_j$  act on  $\mathcal{T}$  via the adjoint action, we can determine their action on  $\kappa$ -Minkowski generators through the duality structure of bicrossproduct Hopf algebras.

When we previously defined the canonical action, we have seen that the action of the ordinary translations on the commutative spacetime coordinates is described by this type of action. So, it appears natural to consider the canonical action as the generalization of the action of the translation generators  $P_\mu$  on  $\kappa$ -Minkowski as well.

Under these assumptions we can show that  $\kappa$ -Minkowski transforms covariantly under the action of the  $\kappa$ -Poincaré generators.

$P_\mu$  acts on its dual space  $\mathcal{T}^*$  (i.e.  $\kappa$ -Minkowski) by canonical action

$$t \triangleright x = \langle x_{(1)}, t \rangle x_{(2)}, \quad t \in \mathcal{T}, x \in \mathcal{T}^*,$$

from which it follows that  $P_\mu$  acts as a derivation on the  $\kappa$ -Minkowski generators:

$$P_\mu \triangleright x_\nu = -i\eta_{\mu\nu}.$$

Their extension to products of the spacetime coordinates is via the covariance condition  $t \triangleright xy = (t_{(1)} \triangleright x)(t_{(2)} \triangleright y)$ , in particular,

$$P_0 \triangleright x_0 x_k = -i x_k \quad P_0 \triangleright x_k x_0 = -i x_k \quad (\text{A.34})$$

$$P_j \triangleright x_0 x_k = (1 \triangleright x_0)(P_j \triangleright x_k) = i \delta_{jk} x_0 \quad (\text{A.35})$$

$$P_j \triangleright x_k x_0 = (P_j \triangleright x_k)(e^{-\lambda P_0} \triangleright x_0) = i(x_0 + i\lambda) \delta_{jk}; \quad (\text{A.36})$$

therefore, the  $\kappa$ -Minkowski commutation relations are invariant under the  $\kappa$ -Poincaré translations.

To derive also the action of  $(M_j, N_j)$  on  $\mathcal{T}^*$  one can use the fact that their right action on  $\mathcal{T}$  dualizes to an action on the left on  $\mathcal{T}^*$  (A.5):

$$\langle P_\mu \triangleleft^{ad} M_k, x \rangle = - \langle P_\mu, M_k \triangleright^{ad} x \rangle \quad (\text{A.37})$$

$$\langle P_\mu \triangleleft^{ad} N_k, x \rangle = - \langle P_\mu, N_k \triangleright^{ad} x \rangle, \quad x \in \mathcal{T}^* \quad (\text{A.38})$$

so, in our case, using (A.33, A.32) and the commutators (1.30), one finds:

$$M_j \triangleright \hat{x}_k = i \epsilon_{jkl} \hat{x}_l, \quad M_j \triangleright \hat{x}_0 = 0, \quad N_j \triangleright \hat{x}_k = i \delta_{jk} \hat{x}_0, \quad N_j \triangleright \hat{x}_0 = i \hat{x}_j \quad (\text{A.39})$$

and, extending these actions via the covariance property of the adjoint action  $h \triangleright xy = (h_{(1)} \triangleright x)(h_{(2)} \triangleright y)$ ,  $h = M_j, N_j$ ,  $x, y \in \mathcal{T}^*$ ,

$$M_j \triangleright \hat{x}_0 \hat{x}_k = i \epsilon_{jkl} \hat{x}_0 \hat{x}_l \quad M_j \triangleright \hat{x}_k \hat{x}_0 = i \epsilon_{jkl} \hat{x}_l \hat{x}_0$$

$$\begin{aligned} N_j \triangleright \hat{x}_0 \hat{x}_k &= (N_j \triangleright \hat{x}_0)(e^{-\lambda P_0} \triangleright \hat{x}_k) + (1 \triangleright \hat{x}_0)(N_j \triangleright \hat{x}_k) - \lambda \epsilon_{jrl} (M_r \triangleright \hat{x}_0)(P_l \triangleright \hat{x}_k) = \\ &= i \hat{x}_j \hat{x}_k + i \delta_{jk} \hat{x}_0^2 \end{aligned}$$

$$\begin{aligned} N_j \triangleright \hat{x}_k \hat{x}_0 &= (N_j \triangleright \hat{x}_k)(e^{-\lambda P_0} \triangleright \hat{x}_0) + (1 \triangleright \hat{x}_k)(N_j \triangleright \hat{x}_0) - \lambda \epsilon_{jrl} (M_r \triangleright \hat{x}_k)(P_l \triangleright \hat{x}_0) = \\ &= i \hat{x}_k \hat{x}_j + i \delta_{jk} \hat{x}_0^2 - \lambda \delta_{jk} \hat{x}_0 \end{aligned}$$

$$\begin{aligned} N_j \triangleright \hat{x}_k \hat{x}_l &= (N_j \triangleright \hat{x}_k)(e^{-\lambda P_0} \triangleright \hat{x}_l) + (1 \triangleright \hat{x}_k)(N_j \triangleright \hat{x}_l) - \lambda \epsilon_{jrs} (M_r \triangleright \hat{x}_k)(P_s \triangleright \hat{x}_l) = \\ &= i \delta_{jk} \hat{x}_0 \hat{x}_l + i \delta_{jl} \hat{x}_k \hat{x}_0 + \lambda (\delta_{lk} \hat{x}_j - \delta_{jk} \hat{x}_l) \end{aligned}$$

$$\begin{aligned} N_j \triangleright \hat{x}_l \hat{x}_k &= (N_j \triangleright \hat{x}_l)(e^{-\lambda P_0} \triangleright \hat{x}_k) + (1 \triangleright \hat{x}_l)(N_j \triangleright \hat{x}_k) - \lambda \epsilon_{jrs} (M_r \triangleright \hat{x}_l)(P_s \triangleright \hat{x}_k) = \\ &= i \delta_{jl} \hat{x}_0 \hat{x}_k + i \delta_{jk} \hat{x}_l \hat{x}_0 + \lambda (\delta_{kl} \hat{x}_j - \delta_{jl} \hat{x}_k) \end{aligned}$$

$$\Rightarrow M_j \triangleright [\hat{x}_k, \hat{x}_0] = i\epsilon_{jkl}[\hat{x}_l, \hat{x}_0] = -\epsilon_{jkl}\lambda\hat{x}_l = M_j \triangleright (i\lambda\hat{x}_k) \quad (\text{A.40})$$

$$\Rightarrow N_j \triangleright [\hat{x}_k, \hat{x}_0] = -\lambda\delta_{jk}\hat{x}_0 = N_j \triangleright (i\lambda\hat{x}_k) \quad (\text{A.41})$$

$$\Rightarrow N_j \triangleright [\hat{x}_k, \hat{x}_l] = i\delta_{jk}[\hat{x}_0, \hat{x}_l] + i\delta_{jl}[\hat{x}_k, \hat{x}_0] + \lambda\delta_{jl}\hat{x}_k - \delta_{jk}\hat{x}_l = 0 = N_j \triangleright 0. \quad (\text{A.42})$$

Thereby the  $\kappa$ -Minkowski commutation relations remain unmodified also under the action of the boost-rotation generators of  $\kappa$ -Poincaré Hopf algebra.

## Appendix B

# Five-dimensional differential calculus

In a noncommutative spacetime it is a highly non-trivial exercise to establish the differential calculus. On  $\kappa$ -Minkowski spacetime one can construct two distinct differential calculi. In chapter 2 we presented the four-dimensional translational invariant calculus proposed by Majid and Oeckl [21], which is however not covariant under the action of the full  $\kappa$ -Poincaré algebra. In this appendix we want to show how the calculus (3.7) bicovariant under the action of the full  $\kappa$ -Poincaré algebra is obtained and why it is necessarily five dimensional.

Let us recall that the  $\kappa$ -Poincaré algebra in the Majid-Ruegg bicrossproduct basis is defined by the commutation relation between the Lorentz and translational sectors:

$$\begin{aligned}
 [P_\mu, P_\nu] &= 0 \\
 [M_j, P_0] &= 0 \\
 [M_j, P_k] &= i\epsilon_{jkl}P_l \\
 [N_j, P_0] &= iP_j \\
 [N_j, P_k] &= i\delta_{jk} \left( \frac{1}{2\lambda}(1 - e^{-2\lambda P_0}) + \frac{\lambda}{2}P^2 \right) - i\lambda P_j P_k
 \end{aligned} \tag{B.1}$$

and the following deformed coproducts:

$$\begin{aligned}
 \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0 \\
 \Delta(P_j) &= P_j \otimes e^{-\lambda P_0} + 1 \otimes P_j \\
 \Delta(M_j) &= M_j \otimes 1 + 1 \otimes M_j \\
 \Delta(N_j) &= N_j \otimes e^{-\lambda P_0} + 1 \otimes N_j - \lambda \epsilon_{jkl} M_k \otimes P_l.
 \end{aligned} \tag{B.2}$$

We have seen in the first chapter that as the  $\kappa$  deformation of Minkowski space we take the dual Hopf algebra of the translation algebra  $\mathcal{T}$  and we denote its generators by  $\hat{x}_\mu$ . Thus  $\kappa$ -Minkowski is defined by:

$$[\hat{x}_i, \hat{x}_j] = 0 \quad [\hat{x}_i, \hat{x}_0] = i\lambda\hat{x}_i, \quad (\text{B.3})$$

$$\Delta\hat{x}_\mu = \hat{x}_\mu \otimes 1 + 1 \otimes \hat{x}_\mu. \quad (\text{B.4})$$

We have shown in Appendix A that the canonical action of translations on  $\kappa$ -Minkowski spacetime is:

$$t \triangleright x = \langle x_{(1)}, t \rangle x_{(2)}, \quad \forall x \in \mathcal{T}^*, \forall t \in \mathcal{T},$$

with the shorthand notation  $\Delta x = \sum x_{(1)} \otimes x_{(2)}$ .

From the bicrossproduct structure of  $\kappa$ -Poincaré we have the action of  $U(\mathfrak{so}(1,3))$  on translations  $\mathcal{T}$ , which, by duality, can be translated into action on the generators of  $\kappa$ -Minkowski:

$$M_i \triangleright \hat{x}_j = i\epsilon_{ijk}\hat{x}_k, \quad M_i \triangleright \hat{x}_0 = 0, \quad N_i \triangleright \hat{x}_j = i\delta_{ij}\hat{x}_0, \quad N_i \triangleright \hat{x}_0 = i\hat{x}_i, \quad (\text{B.5})$$

which generalizes to the whole algebra by the covariance condition:

$$h \triangleright xy = (h_{(1)} \triangleright x)(h_{(2)} \triangleright y), \quad \forall h \in U(\mathfrak{so}(1,3)), x, y \in \mathcal{T}^*. \quad (\text{B.6})$$

The problem we are to solve here is the following. In commutative spacetime positions commute with differentials (one forms). However here we are working with noncommutative spacetime, and thus we cannot assume a priori that positions commute with one forms. Instead, let us take a basis of one forms, which should include differentials  $d\hat{x}_\mu$ , and denote the elements of this basis by  $\chi_a, a = 0, \dots, N, N \geq 4$ . We assume that the commutator  $[\hat{x}_\mu, \chi_a]$  must have the following expansion:

$$[\hat{x}_\mu, \chi_a] = \sum_{\mu, a, b} A_{\mu a}^b \chi_b. \quad (\text{B.7})$$

There are, of course, some consistency conditions for the above relations, which come from the mixed Jacoby identity:

$$[[\hat{x}_\mu, \hat{x}_\nu], \chi_a] + [[\hat{x}_\nu, \chi_a], \hat{x}_\mu] + [[\chi_a, \hat{x}_\mu], \hat{x}_\nu] = 0. \quad (\text{B.8})$$

If we rewrite, for simplicity of notation, the commutation relations (B.3) in a more general form:

$$[\hat{x}_\mu, \hat{x}_\nu] = B_{\mu\nu}^\rho \hat{x}_\rho, \quad (\text{B.9})$$

the relation (B.8) takes the form:

$$A_{\nu c}^a A_{\mu b}^c - A_{\mu c}^a A_{\nu b}^c = B_{\mu\nu}^\rho A_{\rho b}^a. \quad (\text{B.10})$$

Next, expressing  $d\hat{x}_\mu$  as a linear combination of  $\chi_a$ :

$$d\hat{x}_\mu = D_\mu^a \chi_a, \quad (\text{B.11})$$

if we apply the exterior derivative to both sides of (B.9) and we impose the Leibnitz rule, we obtain another restriction:

$$D_\nu^b A_{\mu b}^a - D_\mu^b A_{\nu b}^a = B_{\mu\nu}^\rho D_\rho^a. \quad (\text{B.12})$$

Both relations (B.10) and (B.12) are necessary consistency conditions to determine a bicovariant differential calculus on  $\kappa$ -Minkowski spacetime.

We need now to append these conditions with the covariance requirement, i.e. the condition that both sides of (B.7) transform in the same way under the actions of rotations and boosts. We shall postulate that the action of the Lorentz algebra (B.5)-(B.6) extends to the differential algebra in a natural covariant way, i.e.:

$$h \triangleright (ydx) = (h_{(1)} \triangleright y)(d(h_{(2)} \triangleright x)), \quad h \triangleright (dxy) = (d(h_{(1)} \triangleright x))(h_{(2)} \triangleright y). \quad (\text{B.13})$$

From the above definition and the action (B.5) we obtain for the left side of (B.7) the following identities:

$$N_k \triangleright [\hat{x}_i, d\hat{x}_j] = i\delta_{ki}[\hat{x}_0, d\hat{x}_j] + i\delta_{kj}[\hat{x}_i, d\hat{x}_0] + \lambda(\delta_{kj}d\hat{x}_i - \delta_{ij}d\hat{x}_k), \quad (\text{B.14})$$

$$N_k \triangleright [\hat{x}_0, d\hat{x}_i] = i[\hat{x}_k, d\hat{x}_i] + i\delta_{ki}[\hat{x}_0, d\hat{x}_0] + \lambda\delta_{ki}d\hat{x}_0, \quad (\text{B.15})$$

$$N_k \triangleright [\hat{x}_i, d\hat{x}_0] = i[\hat{x}_i, d\hat{x}_k] + i\delta_{ki}[\hat{x}_0, d\hat{x}_0], \quad (\text{B.16})$$

$$N_k \triangleright [\hat{x}_0, d\hat{x}_0] = i[\hat{x}_k, d\hat{x}_0] + i[\hat{x}_0, d\hat{x}_k] + \lambda d\hat{x}_k, \quad (\text{B.17})$$

$$M_k \triangleright [\hat{x}_i, d\hat{x}_j] = i\epsilon_{kis}[\hat{x}_s, d\hat{x}_j] + i\epsilon_{kjs}[\hat{x}_i, d\hat{x}_s], \quad (\text{B.18})$$

$$M_k \triangleright [\hat{x}_0, d\hat{x}_i] = i\epsilon_{kis}[\hat{x}_0, d\hat{x}_s], \quad (\text{B.19})$$



$$M_k \triangleright [\hat{x}_i, d\hat{x}_0] = i\epsilon_{kis}[\hat{x}_s, d\hat{x}_0], \quad (\text{B.20})$$

$$M_k \triangleright [\hat{x}_0, d\hat{x}_0] = 0; \quad (\text{B.21})$$

we will demonstrate just the first of the identities above, the others follows analogously:

$$\begin{aligned} N_k \triangleright [\hat{x}_i, d\hat{x}_j] &= N_k \triangleright (\hat{x}_i d\hat{x}_j - d\hat{x}_j \hat{x}_i) = \\ &= (N_k \triangleright \hat{x}_i) d(e^{-\lambda P_0} \triangleright \hat{x}_j) + 1 \triangleright \hat{x}_i (d(N_k \triangleright \hat{x}_j)) - \lambda \epsilon_{klm} M_l \triangleright \hat{x}_i (d(P_m \triangleright \hat{x}_j)) + \\ &\quad - d(N_k \triangleright \hat{x}_j) e^{-\lambda P_0} \triangleright \hat{x}_i - d(1 \triangleright \hat{x}_j) (N_k \triangleright \hat{x}_i) + d(\lambda \epsilon_{klm} M_l \triangleright \hat{x}_j) P_m \triangleright \hat{x}_i = \\ &= i\delta_{ki} \hat{x}_0 d\hat{x}_j + i\delta_{kj} \hat{x}_i d\hat{x}_0 - \lambda \epsilon_{kli} \epsilon_{ljr} d\hat{x}_r - i\delta_{ki} d\hat{x}_0 \hat{x}_i - i\delta_{ki} d\hat{x}_j \hat{x}_0 = \\ &= i\delta_{ki} [\hat{x}_0, d\hat{x}_j] + i\delta_{kj} [\hat{x}_i, d\hat{x}_0] + \lambda (\delta_{kj} \delta_{ir} - \delta_{ij} \delta_{kr}) d\hat{x}_r = \\ &= i\delta_{ki} [\hat{x}_0, d\hat{x}_j] + i\delta_{kj} [\hat{x}_i, d\hat{x}_0] + \lambda (\delta_{kj} d\hat{x}_i - \delta_{ij} d\hat{x}_k). \end{aligned}$$

Transforming also the right side of (B.7) under the action of rotations and boosts (with  $\chi_\mu = d\hat{x}_\mu$ ) and imposing the equivalence with the identities (B.14)-(B.21) we obtain a system of linear equations for the coefficients  $A_{\mu\nu}^\rho$  which we can solve, reminding that the consistency conditions (B.10) and (B.12) must be satisfied.

Now, if we consider only 4D bicovariant calculi, it appears that the solution is unique and gives us the following relations:

$$[\hat{x}_i, d\hat{x}_j] = i\delta_{ij} \lambda d\hat{x}_0 \quad [\hat{x}_i, d\hat{x}_0] = i\lambda d\hat{x}_i \quad (\text{B.22})$$

$$[\hat{x}_0, d\hat{x}_j] = 0 \quad [\hat{x}_0, d\hat{x}_0] = 0, \quad (\text{B.23})$$

which, however, do not define a differential calculus as they fail to obey the condition (B.10). Therefore we conclude that there not exist a four-dimensional bicovariant differential calculus on  $\kappa$ -Minkowski spacetime.

Thus we see that the basis of one-forms of the bicovariant differential calculus is indeed  $\chi_a = (d\hat{x}_\mu, d\hat{x}_4)$ . Since  $d\hat{x}_4$  does not carry the spacetime index, it must be invariant under the action of the Lorentz generators

$$N_i \triangleright d\hat{x}_4 = 0 \quad M_i \triangleright d\hat{x}_4 = 0. \quad (\text{B.24})$$

Now, solving the system of linear equations (B.14)-(B.21) and imposing the consistency conditions (B.10) and (B.12), one finds that the commutation relations between the coordinates  $\hat{x}_\mu$  and all the generating one-forms  $d\hat{x}_\mu, d\hat{x}_4$  are the following:

$$[\hat{x}_0, d\hat{x}_4] = i\lambda d\hat{x}_0 \quad [\hat{x}_0, d\hat{x}_0] = i\lambda d\hat{x}_4 \quad [\hat{x}_0, d\hat{x}_i] = 0 \quad (\text{B.25})$$

$$[\hat{x}_i, d\hat{x}_4] = [\hat{x}_i, d\hat{x}_0] = -i\lambda d\hat{x}_i \quad [\hat{x}_i, d\hat{x}_j] = i\lambda\delta_{ij}(d\hat{x}_4 - d\hat{x}_0). \quad (\text{B.26})$$

These relations define a five-dimensional bicovariant differential calculus on  $\kappa$ -Minkowski spacetime.



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